New Formulation of Anomaly, Anomaly Formula and Graphical Representation *

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Abstract

A general approach to anomaly in quantum field theory is newly formulated by use of the propagator theory in solving the heat-kernel equation. We regard the heat-kernel as a sort of the point-splitting regularization in the space(-time) manifold. Fujikawa's general standpoint that the anomalies come from the path-integral measure is taken. We obtain some useful formulae which are valid for general anomaly calculation. They turn out to be the same as the 1-loop counter-term formulae except some important total derivative terms. Various anomalies in 2 and 4 dimensional theories are systematically calculated. Some important relations between them are concretely shown. As for the representation of general (global SO(n)) tensors, we introduce a graphical one . It makes the tensor structure of invariants very transparent and makes the tensor calculation so simple.

1 Introduction

A symmetry, in the field theory, imposed at the classical level sometimes cannot survive at the quantum level due to the appearance of anomaly. It began to be noticed around 1950 by Fukuda and Miyamoto [1] and was clearly recognized as the triangle anomaly around 1970 by Adler [2] and Bell and Jackiw [3]. The basic origin at that time was considered as the linear divergence and its regularization ambiguity in a triangle diagram. The fresh standpoint on the anomaly was brought by Fujikawa [4, 5] around 1980. He regarded the anomaly, including the Weyl anomaly, as the Jacobian in the change of field variable in the path-integral formalism. In the evaluation Fujikawa paid close attention to the definition of the path-integral measure. He also pointed out the connection with the Atiyah-Singer theorem. The anomaly was

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generalized to gravitational theories in the general dimension by Alvarez-Gaumé and Witten [6], especially they found the pure gravitational anomaly. They exploited the knowledge of differential geometry to obtain the explicit form of the chiral U(1) anomaly in 2k dim gravitational theories. In mid 80's the geometrical aspect of the anomaly was much examined[7].

In the present paper, special emphasis is on the Weyl anomaly which is different from other anomalies in the following points.

1. Weyl anomaly appears in general theories because it is directly related to the trace part of the energy-momentum tensor. It is essentially given by the β-function, which determines the scaling properties of a theory.

2. Weyl anomaly, at present, does not seem to be understood only by the global geometrical (topological) analysis. This is related to the fact of item 1.

3. Quantum effect of gauge or gravitational fields is equally important to that of matter fields.

4. The Weyl anomaly is the problem of the real part of the effective action.

Therefore we can expect the Weyl anomaly contains richer dynamical information than other anomalies.

Motivated by this expectation, we newly formulate the general anomaly problem. It is based on (the coordinate version of) the *propagator approach* in the ordinary perturbative field theory[8]. All explanation is done by the familiar field theory language. We take the heat-kernel regularization for the ultraviolet divergences. General formulae of anomalies are obtained. Many applications are presented. Weyl anomalies, chiral anomalies(in the flat and curved space), local Lorentz anomaly and gravitational anomaly are explicitly derived. Practical usefulness is stressed.

Generally anomaly terms (especially Weyl anomaly terms), including their coefficients, are fixed by calculating (1-loop) quantum fluctuation in the perturbative expansion. The explicit calculation becomes more and more complicated as we increase the space(-time) dimension. In n-dim gravitational theories, we must treat complicated higher-rank global SO(n) covariants and invariants such as $\partial_\mu\partial_\nu h_{\alpha\beta},\ \partial_\mu\partial_\nu h_{\lambda\sigma}\cdot\partial_\mu\partial_\nu h_{\lambda\sigma}$, etc. in the weak-gravity expansion: $g_{\alpha\beta}=\delta_{\alpha\beta}+h_{\alpha\beta},\ |h_{\alpha\beta}|\ll 1,\ (\alpha,\beta,\cdots=1,2,\cdots,n)$. In order to get rid of the obstacle, we introduce a graphical representation to treat those terms systematically. The representation makes it so easy to list up all independent terms. Although the present paper deals with $n=2$ and 4 cases only, in order to clearly show the usefulness, it is applicable to higher-dimensional cases.

The anomaly terms are often compared with the counter-terms. The former is independent of gauge whereas the latter is not. The former is related with the consistency of the theory whereas the latter is related with the renormalization of the theory. Although both come from the ultra-violet divergences, their most appropriate regularizations are different: the anomaly terms are commonly calculated by use of Pauli-Villars regularization (or its variants) whereas the counter-terms are usually calculated by use of the dimensional regularization. We will obtain some direct

connection between the two quantities through the anomaly formula calculated by the heat-kernel regularization. The formula is very powerful as in the case of the counter-term formula by 'tHooft[9]. It is demonstrated that various anomalies of various theories are derived by the formulae.

The content is organized as follows. In Sec.2 we explain the present formulation of anomaly taking a simple model of the scalar-gravity system in n dimension. It is based on the propagator theory. The heat-kernel regularization and a graphical representation are taken. We obtain 2 dim and 4 dim anomaly formula in Sec.3 and 4 respectively. In Sec.5, generalization to fermion-gravity system is presented. The 2 dim anomaly formula is applied to some interesting anomalies of 2 dim theories in Sec.6. Relations between different anomalies are explained. In Sec.7 chiral anomalies in flat and gravitational theories are obtained by the 4 dim anomaly formula. The relation between 2 dim gravitational anomaly and 4 dim chiral gravitational anomaly is also explained. We conclude in Sec.8. App.A is prepared for the present notation and some useful formula. We explain the graphical representation in detail in App.B and C. App.B is for the global SO(n) covariants and invariants such as $\partial_\alpha\partial_\beta h_{\mu\nu}$, which appear in concrete substitution of the (weak-field) expanded terms into the formulae. App.C is for those which appear as general terms in the anomaly formulae.

2 New Formulation of Anomaly and Heat-Kernel Regularization

2.1 Anomaly and Jacobian Factor

Let us explain the present formulation of anomalies taking a simple example : Weyl anomaly in n-dim Euclidean gravity-scalar coupled system.

$$\mathcal{L}[g_{\mu\nu},\phi]=\sqrt{g}(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}qR\phi^2) \quad ,$$
$$q=-\frac{n-2}{4(n-1)} \quad , \tag{1}$$

where $g_{\mu\nu}$ and ϕ are the metric field and the scalar field. This Lagrangian is invariant under the local Weyl transformation:

$$g^{\mu\nu}(x)'=e^{2\alpha(x)}g^{\mu\nu}(x) \quad , \quad \phi(x)'=e^{\frac{n-2}{2}\alpha(x)}\phi(x) \quad , \quad \tilde{\phi}(x)'=e^{-\alpha(x)}\tilde{\phi}(x) \quad , \tag{2}$$

where $\alpha(x)$ is the parameter of the local Weyl transformation, $g=det\ g_{\mu\nu}$, and we introduce $\tilde{\phi}\equiv\sqrt[4]{g}\ \phi$ for the measure $\mathcal{D}\tilde{\phi}$ to be general coordinate invariant [10]. The partition function ,on the external gravitational field $g_{\mu\nu}$, is given by

$$Z[g_{\mu\nu}]=\int\mathcal{D}\tilde{\phi}\ exp\{\ -S[g_{\mu\nu},\phi]\ \} \quad , \quad S[g_{\mu\nu},\phi]=\int d^nx\mathcal{L}[g_{\mu\nu},\phi] \quad . \tag{3}$$

Now let us see the response of $Z[g_{\mu\nu}]$ under the Weyl transformation.

$$Z[g'_{\mu\nu}]=\int\mathcal{D}\tilde{\phi}'exp\{\ -\int d^nx\ \mathcal{L}[g'_{\mu\nu},\phi']\ \}$$
$$=\int\mathcal{D}\tilde{\phi}(x)\ det\frac{\partial\tilde{\phi}'(y)}{\partial\tilde{\phi}(x)}exp\{\ -\int d^nx\mathcal{L}[g_{\mu\nu},\phi]\ \} \quad . \tag{4}$$

Comparing (4) with (3), we see the variation of $Z[g_{\mu\nu}]$ comes from the Jacobian factor in the Weyl transformation of integration measure [4]. The Jacobian is formally written as

$$J \equiv det\frac{\partial\tilde{\phi}'(y)}{\partial\tilde{\phi}(x)} = det\ (e^{-\alpha(x)}\delta^n(x-y)\ )$$
$$= exp(-Tr\ [\alpha(x)\delta^n(x-y)] + O(\alpha^2)) \quad . \qquad (5)$$

In order to regularize the delta function $\delta^n(x-y)$, we introduce the following quantity.

$$G(x,y;t) \equiv < x|e^{-t\vec{D}}|y> \quad , \quad t>0 \quad ,$$
$$\vec{D}_x \equiv \sqrt[4]{g}(-\nabla_x^2 + qR(x))\frac{1}{\sqrt[4]{g}} \quad , \qquad (6)$$

where t will be regarded as a regularization parameter and is called Schwinger's *proper time* [11]. The operator $\vec{D}_x$ is the hermitian differential (energy) operator which appears in the field equation for $\tilde{\phi}$:

$$\frac{\delta S}{\delta\phi(x)} = \sqrt{g}(-\nabla^2 + qR)\phi = \sqrt[4]{g}\vec{D}_x\tilde{\phi}(x) = 0 \quad ,$$
$$\vec{D} = \vec{D}^\dagger \quad . \qquad (7)$$

Here we note the following general features.

- The physical dimension of the parameter t is (Length)^2.
- The bra- and ket-vector $(<x|, |y>)$ are rather symbolically introduced by Schwinger [11]. They can be more precisely defined as follows[4] . Let $f_i(x)$ be the complete ortho-normal eigen-functions of the differential (energy) operator $\vec{D}$:

  $$\vec{D}_x f_i(x) = \lambda_i f_i(x),\ \int d^n x f_i(x)^\dagger f_j(x) = \delta_{ij},\ \sum_i f_i(x) f_i^\dagger(y) = \delta^n(x-y)I \quad , \qquad (8)$$

  where I is some identity (constant) matrix. We define the bra and ket vectors, following Dirac[12], by

  $$f_i(x) \equiv < x|i> \quad , \quad f_i^\dagger(x) \equiv < i|x> \quad . \qquad (9)$$

  The orthogonality and the completeness conditions (8) are realized by the following requirements:

  $$\begin{array}{lll} \text{Orthonormal Condition} & <i|j>=\delta_{ij} \quad , & \int d^n x|x><x| = I \quad , \\ \text{Completeness Condition} & <x|y>=\delta^n(x-y) \quad , & \sum_i |i><i| = I \quad . \end{array} \qquad (10)$$

  The hermiticity of the operator $\vec{D}$ (or the matrix: $D_{ij} \equiv \int d^n x f_i^\dagger(x)\vec{D}_x f_j(x) = \lambda_i\delta_{ij}$) means real eigenvalues: $\lambda_i = \lambda_i^*$. $G(x,y;t)$ satisfies

  $$G(x,y;t) =< x|e^{-t\vec{D}}(\sum_i |i><i|)|y> = \sum_i e^{-t\lambda_i} f_i(x) f_i^\dagger(y) \quad ,$$
  $$(\frac{\partial}{\partial t} + \vec{D}_x)G(x,y;t) = 0 \quad , \quad G(x,y;t)(\frac{\overleftarrow{\partial}}{\partial t} + \overleftarrow{\vec{D}}_y^\dagger) = 0 \quad , \qquad (11)$$

where $\overleftarrow{D}_x$ means it operates on the left . The last two equations show the symmetric property of $G(x,y)$ with respect to x and y. This property is desirable as the regularization of $\delta^n(x-y)$. We can regard the equations (11) as the precise definition of $G(x,y;t)$ introduced in (6).

- From the result of the previous item, the positivity of the eigenvalues λ_i is required for the well-defined-ness of $G(x,y;t)$ for $t \rightarrow +\infty$. (This is important for the evaluation of the effective action.)

- In order for $G(x,y;t)$ to properly regularize $\delta^n(x-y)$, $\vec{D}_x$ must, at least, satisfy the conditions: 1) hermiticity; 2) positivity of eigenvalues. For the correspondence to the anomaly calculation based on the ordinary (perturbative) quantization, we usually take , as $\vec{D}_x$, a differential operator which appears in the field equation. $\vec{D}_x$ is usually an elliptic differential operator. When the differetial operator in the field equation does not satisfy the above conditions, ambiguity manifestly appears in the choice of $\vec{D}_x$ and the anomaly terms are not unique. See Sec.6 for such an example.

In the next subsection, we will see that $G(x,y;t)$ regularizes $\delta^n(x-y)$ (see eq.(13)).

2.2 Perturbative Solution of $G(x,y;t)$ and Heat Equation

$G(x,y;t)$ satisfies the following differential equation.

$$(\frac{\partial}{\partial t}+\vec{D}_x)G(x,y;t)=0 \quad , \quad t>0 \quad . \tag{12}$$

We solve this equation with the following initial condition.

$$\lim_{t\rightarrow +0} G(x,y;t)=\delta^n(x-y) \quad . \tag{13}$$

This equation expresses the regularization of the delta function $\delta^n(x-y)$ where the proper time t plays the role of the regularization parameter [13, 14]. For later general use, we write (12) and (13) in a more general form as,

$$\begin{aligned} &(\frac{\partial}{\partial t}\delta^{ij}+\vec{D}_x^{ij})G^{jk}(x,y;t)=0 \quad , \quad t>0 \quad . \\ &\lim_{t\rightarrow +0} G^{ij}(x,y;t)=\delta^n(x-y)\delta^{ij} \quad , \quad i,j=1,2,\cdots N \quad , \end{aligned} \tag{14}$$

where i and j are the field suffixes , such as a fermion suffix and a vector suffix. In the present example $N=1$.

For a general theory with the derivative couplings up to the second order, the operator $\vec{D}_x^{ij}$ can be always written as

$$\begin{aligned} \vec{D}_x^{ij} &= -\delta_{\mu\nu}\delta^{ij}\partial_\mu\partial_\nu-\vec{V}^{ij}(x) \quad , \\ \vec{V}^{ij}(x) &\equiv W_{\mu\nu}^{ij}(x)\partial_\mu\partial_\nu+N_\mu^{ij}\partial_\mu+M^{ij} \quad , \end{aligned} \tag{15}$$

where $W^{ij}_{\mu\nu}$, N^{ij}_μ and M^{ij} are external fields (background coefficient fields). In the present example, the above quantities are explicitly written as

$$
\begin{aligned}
\vec{V}(x) &= \sqrt[4]{g}(\nabla^\mu\nabla_\mu - qR)\frac{1}{\sqrt[4]{g}} - \delta_{\mu\nu}\partial_\mu\partial_\nu \quad , \\
W_{\mu\nu} &= g^{\mu\nu} - \delta_{\mu\nu} = -h_{\mu\nu} + h_{\mu\lambda}h_{\lambda\nu} + O(h^3) \quad , \\
N_\lambda &= -g^{\mu\nu}\Gamma^\lambda_{\mu\nu} - g^{\lambda\mu}\Gamma^\nu_{\mu\nu} = -\partial_\mu h_{\lambda\mu} + O(h^2) \quad , \\
M &= -qR + \frac{1}{4}g^{\mu\nu}\{\Gamma^\lambda_{\mu\lambda}\Gamma^\sigma_{\nu\sigma} + 2\Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - 2\partial_\nu\Gamma^\lambda_{\mu\lambda}\} \\
&= -q(\partial^2 h - \partial_\alpha\partial_\beta h_{\alpha\beta}) - \frac{1}{4}\partial^2 h + O(h^2) \quad ,
\end{aligned}
\tag{16}
$$

where $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, $h \equiv h_{\mu\mu}$. The usage of general coefficients $W^{ij}_{\mu\nu}, N^{ij}_\mu$ and M^{ij} ,instead of their concrete contents, makes it possible to obtain a general formula for anomalies[15, 16] . Let us solve the differential equation (14) perturbatively for the case of weak external fields $(W^{ij}_{\mu\nu}, N^{ij}_\mu, M^{ij})$. (In the present example, this corresponds to the perturbation around the *flat* space.) The differential equation (14) becomes

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \partial^2)G^{ij}(x,y;t) = \vec{V}^{ik}(x)G^{kj}(x,y;t) \quad , \\
&\qquad \partial^2 \equiv \delta_{\mu\nu}\partial_\mu\partial_\nu = \sum_{\mu=1}^{n}(\frac{\partial}{\partial x^\mu})^2 \quad .
\end{aligned}
\tag{17}
$$

In the following we suppress the field suffixes $i, j, \cdots$ and take the matrix notation. This equation (17) is the n-dim heat equation with the small perturbation $\vec{V}$. We prepare two quantities in order to obtain the solution. (This approach is popular in the perturbative quantum field theory under the name of *propagator approach*[8]. In [8] the momentum representation is taken, which is to be compared with the coordinate one of the present approach. The Weyl anomaly in the string theory was analyzed in this approach by Alvarez[17].)

i) Heat Equation

The heat equation:

$$
(\frac{\partial}{\partial t} - \partial^2)G_0(x,y;t) = 0 \quad , \quad t > 0 \tag{18}
$$

has the solution

$$
\begin{aligned}
G_0(x,y;t) = G_0(x-y;t) &= \int \frac{d^n k}{(2\pi)^n} exp\{-k^2 t + ik^\mu(x-y)^\mu\} \ I_N \\
&= \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} \ I_N \quad , \quad k^2 \equiv \sum_{\mu=1}^{n}(k^\mu)^2 \quad ,
\end{aligned}
\tag{19}
$$

where I_N is the identity matrix of the size $N \times N$. G_0 satisfies the initial condition: $\lim_{t\to+0} \ G_0(x-y;t) = \delta^n(x-y) \ I_N$. We define

$$
G_0(x,y;t) = 0 \text{ for } t \le 0 \quad . \tag{20}
$$

ii) Heat Propagator

The heat equation with the delta-function source defines the heat propagator.

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \partial^2) S(x,y;t-s) &= \delta(t-s)\delta^n(x-y)\ I_N \quad , \\
S(x,y;t) = S(x-y;t) = \int \frac{d^n k}{(2\pi)^n}\frac{dk^0}{2\pi} & \frac{exp\{-ik^0 t + ik\cdot(x-y)\}}{-ik^0+k^2}\ I_N \\
&= \theta(t) G_0(x-y;t) \quad , \\
k^2 \equiv \sum_{\nu=1}^{\mu} k^\mu k^\mu \quad , & \quad k\cdot x \equiv \sum_{\mu=1}^{n} k^\mu x^\mu \quad .
\end{aligned} \tag{21}
$$

$\theta(t)$ is the *step function* defined by : $\theta(t) = 1$ for $t > 0$, $\theta(t) = 0$ for $t < 0$. $S(x-y;t)$ satisfies the initial condition: $\lim_{t\to+0} S(x-y;t) = \delta^n(x-y)\ I_N$ and $S(x,y;t) = 0$ for $t \leq 0$.

Now the formal solution of (17) with the initial condition (14) is given by

$$
G(x,y;t) = G_0(x-y;t) + \int d^n z \int_{-\infty}^{\infty} ds\ S(x-z;t-s)\vec{V}(z)G(z,y;s) \quad . \tag{22}
$$

$G(x,y;t)$ appears in both sides above. We can iteratively solve (22) as

$$
\begin{aligned}
G(x,y;t) &= G_0(x-y;t) + \int S\vec{V}G_0 + \int S\vec{V}\int S\vec{V}G_0 + \cdots \quad , \\
G_1(x,y;t) &\equiv \int S\vec{V}G_0 = \int d^n z ds S(x-z;t-s)\vec{V}(z)G_0(z-y;s) \\
&= \int d^n z \int_0^t ds G_0(x-z;t-s)\vec{V}(z)G_0(z-y;s) \quad , \\
G_2(x,y;t) &\equiv \int S\vec{V}\int S\vec{V}G_0 = \int d^n z' ds' S(x-z';t-s')\vec{V}(z') \\
&\quad \times \int d^n z ds S(z'-z;s'-s)\vec{V}(z)G_0(z-y;s) \\
&= \int d^n z' \int_0^t ds' G_0(x-z';t-s')\vec{V}(z') \\
&\quad \times \int d^n z \int_0^{s'} ds G_0(z'-z;s'-s)\vec{V}(z)G_0(z-y;s) \quad .
\end{aligned} \tag{23}
$$

Higher-order terms are similarly obtained. Generally, in n-dim, the terms up to $G_{n/2}$ are practically sufficient for the anomaly calculation[18]. The trace-part of (5) is given by putting $x = y$ in the above equations.

$$
\begin{aligned}
G(x,x;t) &= G_0(0;t) + G_1(x,x;t) + G_2(x,x;t) + \cdots \quad , \\
G_0(0;t) &= \frac{1}{(4\pi t)^{n/2}}\ I_N \quad , \\
G_1(x,x;t) &\equiv \int S\vec{V}G_0\Big|_{x=y} \\
&= \frac{1}{(4\pi)^n t^{(n/2)-1}} \int d^n w \int_0^1 dr \frac{1}{\{(1-r)r\}^{n/2}} e^{-\frac{w^2}{4(1-r)}}\ \vec{V}(x+\sqrt{t}w) e^{-\frac{w^2}{4r}} , \\
\vec{V}(x+\sqrt{t}w) &= \frac{1}{t}W_{\mu\nu}(x+\sqrt{t}w)\frac{\partial}{\partial w^\mu}\frac{\partial}{\partial w^\nu} + \frac{1}{\sqrt{t}}N_\mu(x+\sqrt{t}w)\frac{\partial}{\partial w^\mu} + M(x+\sqrt{t}w) \ ,
\end{aligned} \tag{24}
$$

where we introduce some scaled integration variables which are dimension-less: $r = \frac{s}{t}$, $w^\mu = (z-x)^\mu/\sqrt{t}$. Furthermore

$$
\begin{aligned}
& G_2(x,x;t) \equiv \int S\vec{V}\int S\vec{V}G_0\Big|_{x=y} \\
= & \frac{1}{(4\pi)^{3n/2}}\frac{1}{t^{(n/2)-2}}\int d^nv\int d^nu\int_0^1 dk\int_0^k dl\frac{1}{\{(1-k)(k-l)l\}^{n/2}}e^{-\frac{v^2}{4(1-k)}} \quad (25)\\
& \times\vec{V}(x+\sqrt{t}v)e^{-\frac{(v-u)^2}{4(k-l)}}\vec{V}(x+\sqrt{t}u)e^{-\frac{u^2}{4l}} \quad ,
\end{aligned}
$$

where $k = s'/t$, $l = s/t$, $v^\mu = (z'-x)^\mu/\sqrt{t}$, $u^\mu = (z-x)^\mu/\sqrt{t}$.

Further analysis will be done for each dimension.

3 Anomaly Formula in 2 Dimension

We consider the simplest case of the dimension n=2. From (24), we obtain

$$
G_0(0;t) = \frac{1}{4\pi t}\ I_N \quad . \qquad (26)
$$

From (24),we obtain

$$
\begin{aligned}
G_1(x,x;t) &= \frac{1}{(4\pi)^2}\int d^2w\int_0^1 dr\frac{1}{(1-r)r}e^{-\frac{w^2}{4(1-r)}} \\
\times\{\ (\frac{1}{t}W_{\mu\nu}(x) &+ \frac{1}{\sqrt{t}}w^\lambda\partial_\lambda W_{\mu\nu}(x) + \frac{1}{2}w^\lambda w^\sigma\partial_\lambda\partial_\sigma W_{\mu\nu}(x))\frac{\partial}{\partial w^\mu}\frac{\partial}{\partial w^\nu} \\
&+(\frac{1}{\sqrt{t}}N_\nu(x) + w^\lambda\partial_\lambda N_\mu(x))\frac{\partial}{\partial w^\mu} + M(x)\ \}\ e^{-\frac{w^2}{4r}} + O(t) \qquad (27)\\
&= \frac{1}{4\pi}\{\ -\frac{1}{2t}W_{\mu\mu}(x) - \frac{1}{12}\partial^2 W_{\mu\mu}(x) + \frac{1}{3}\partial_\mu\partial_\nu W_{\mu\nu}(x) \\
&\qquad -\frac{1}{2}\partial_\mu N_\mu(x) + M(x)\ \} + O(t) \quad ,
\end{aligned}
$$

where we consider the limit: $t \to +0$. The t^0-part of $G_1(x,x;t)$, written in terms of $W_{\mu\nu}, N_\mu, M$ in the final expression of (27),is the anomaly formula in 2 dimension. The higher-order ones $G_n(x,x;t)$ $(n \geq 2)$ gives higher-terms (with respect to external fields or $h_{\mu\nu}$) and is practically not necessary to fix the anomalies[18].

We apply the above formula to the present example: Weyl anomaly of (1) with $n = 2$. (Note that $q = 0$ in (1) for the dimension $n = 2$.) Using the expanded expression for W, N, M ,(16),we obtain

$$
\begin{aligned}
G(x,x;t) &= G_0(0;t) + G_1(x,x;t) + O(h^2) \\
&= \frac{1}{4\pi t}(1+\frac{h}{2}) - \frac{1}{24\pi}(\partial^2 h - \partial_\mu\partial_\nu h_{\mu\nu}) + O(h^2) + O(t) \qquad (28)\\
&= \frac{1}{4\pi t}(\sqrt{g} + O(h^2)) - \frac{\sqrt{g}}{24\pi}(R + O(h^2)) + O(t) \quad ,
\end{aligned}
$$

where we use the requirement: $\mathrm{Tr}[\alpha(x)\delta^2(x-y)] = \lim_{t\rightarrow+0}\lim_{y\rightarrow x}\int d^2x\alpha(x)G(x,y;t)$ $= \lim_{t\rightarrow+0}\int d^2x\alpha(x)G(x,x;t)$ = general coordinate invariant . (This can be manifestly shown by the normal coordinate expansion[19, 20, 21].) Therefore the Weyl anomaly is given by

$$
\begin{aligned}
\text{Weyl Anomaly} &= \frac{\delta}{\delta\alpha(x)}\ ln\ J\Big|_{\alpha=0} = \frac{\delta}{\delta\alpha(x)}[-\mathrm{Tr}\ \alpha(x)\delta^2(x-y)] \\
&= -\lim_{t\rightarrow+0}G(x,x;t) = \sqrt{g}[-\frac{1}{4\pi t}+\frac{R}{24\pi}] \quad . \qquad (29)
\end{aligned}
$$

The first term of the final form above is divergent and is renormalized by the cosmological term.

The Weyl anomaly in 2 dim fermion- gravity system will be considered in Sect.6 where other important anomalies are also examined.

4 Anomaly Formula in 4 Dimension

In this section we will obtain a formula for anomalies in 4 dim.

4.1 $G_0(0;t), G_1(x,x;t)$

From (19) with $n=4$, we obtain

$$
G_0(0;t) = \frac{1}{(4\pi t)^2}\ I_N \quad . \qquad (30)
$$

From (24),

$$
G_1(x,x;t) = \frac{1}{(4\pi)^4 t}\int d^4w\int_0^1 dr\frac{1}{\{(1-r)r\}^2}e^{-\frac{w^2}{4(1-r)}}\vec{V}(x+\sqrt{t}w)e^{-\frac{w^2}{4r}} \quad . \qquad (31)
$$

We notice t^0-terms only contribute to the anomalies. They correspond to log-divergent terms in the effective action: $\Gamma = \int_0^\infty \frac{dt}{t}G(x,x;t)$. (t^{-m}-terms ($m = 1,2,\cdots$) contribute to power-divergences, t^{+m}-terms vanish as $t\rightarrow +0$.) External fields, $W_{\mu\nu}, N_\mu$ and M in $\vec{V}(x+\sqrt{t}w)$ are expanded around $t=0$ as,

$$
\begin{aligned}
W_{\mu\nu}(x+\sqrt{t}w) &= W_{\mu\nu}(x)+\sqrt{t}w^\alpha\partial_\alpha W_{\mu\nu}(x)+\frac{t}{2}w^\alpha w^\beta\partial_\alpha\partial_\beta W_{\mu\nu}(x)+\cdots \\
N_\mu(x+\sqrt{t}w) &= N_\mu(x)+\sqrt{t}w^\alpha\partial_\alpha N_\mu(x)+\frac{t}{2}w^\alpha w^\beta\partial_\alpha\partial_\beta N_\mu(x)+\cdots \qquad (32) \\
M(x+\sqrt{t}w) &= M(x)+\sqrt{t}w^\alpha\partial_\alpha M(x)+\frac{t}{2}w^\alpha w^\beta\partial_\alpha\partial_\beta M(x)+\cdots \quad .
\end{aligned}
$$

Then we can pick up t^0-part of (31) as follows.

$$
\begin{aligned}
G_1(x,x;t)|_{t^0} &= \frac{1}{(4\pi)^4}\int d^4w\int_0^1 dr\frac{1}{\{(1-r)r\}^2}e^{-\frac{w^2}{4r(1-r)}} \\
&\quad\times\{\ \frac{1}{4!}w^\lambda w^\sigma w^\tau w^\omega\partial_\lambda\partial_\sigma\partial_\tau\partial_\omega W_{\mu\nu}(x)(-\frac{\delta_{\mu\nu}}{2r}+\frac{w^\mu w^\nu}{4r^2})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3!}w^\lambda w^\sigma w^\tau \partial_\lambda\partial_\sigma\partial_\tau N_\mu(x)(-\frac{w^\mu}{2r}) + \frac{1}{2}w^\lambda w^\sigma \partial_\lambda\partial_\sigma M(x)\ \} \quad (33)\\
= \quad &\frac{1}{(4\pi)^2 4!}\{\ -\frac{1}{5}\partial^2\partial^2 W_{\mu\mu}(x) + \frac{6}{5}\partial^2\partial_\mu\partial_\nu W_{\mu\nu}(x)\\
&\qquad -2\partial^2\partial_\mu N_\mu(x) + 4\partial^2 M(x)\ \} \quad .
\end{aligned}
$$

This is one part of the 4 dim anomaly formula. Let us derive the other part.

4.2 $G_2(x,x;t)$

From (25), we have

$$
\begin{aligned}
G_2(x,x;t) = &\frac{1}{(4\pi)^6}\int d^4v d^4u \int_0^1 dk \int_0^k dl \frac{1}{\{(1-k)(k-l)l\}^2} e^{-\frac{1}{4}(\frac{v^2}{1-k}+\frac{(v-u)^2}{k-l}+\frac{u^2}{l})}\\
&\times\{\ \frac{1}{t}W_{\mu\nu}(x+\sqrt{t}v)(-\frac{\delta_{\mu\nu}}{2(k-l)} + \frac{(v-u)^\mu(v-u)^\nu}{4(k-l)^2})\\
&+\frac{1}{\sqrt{t}}N_\mu(x+\sqrt{t}v)(-\frac{(v-u)^\mu}{2(k-l)}) + M(x+\sqrt{t}v)\ \} \quad (34)\\
&\times\{\ \frac{1}{t}W_{\lambda\sigma}(x+\sqrt{t}u)(-\frac{\delta_{\lambda\sigma}}{2l}+\frac{u^\lambda u^\sigma}{4l^2}) + \frac{1}{\sqrt{t}}N_\lambda(x+\sqrt{t}u)(-\frac{u^\lambda}{2l}) + M(x+\sqrt{t}u)\ \} \quad .
\end{aligned}
$$

For the comparison with the (1-loop) counter-term formula, we first consider the case of 'flat' space: $W_{\mu\nu}=0$. In this case the t^0-part of (34) is given as

$$
\begin{aligned}
G_2^{flat}(x,x;t)|_{t^0} \quad = \quad &\frac{1}{(4\pi)^2}[\ -\frac{1}{12}\partial_\nu(\partial_\nu N_\mu\cdot N_\mu)\\
&+\frac{1}{48}(\partial_\mu N_\nu - \partial_\nu N_\mu)^2 + \frac{1}{2}(M-\frac{1}{2}\partial_\mu N_\mu)^2\ ] \quad . \qquad (35)
\end{aligned}
$$

Combining the above result and (33) for $W_{\mu\nu}=0$, $G^{flat}(x,x;t)|_{t^0}$ is expressed, up to the present approximation, as

$$
\begin{aligned}
G^{flat}(x,x;t)|_{t^0} = &\frac{1}{(4\pi)^2}[\ \frac{1}{6}\partial^2(M-\frac{1}{2}\partial_\mu N_\mu) - \frac{1}{12}\partial_\nu(\partial_\nu N_\mu\cdot N_\mu)\ ]\\
&+\Delta\mathcal{L}_{tH} + O((N,M)^3) \quad ,\\
\Delta\mathcal{L}_{tH} \equiv &\frac{1}{(4\pi)^2}[\ \frac{1}{48}(\partial_\mu N_\nu - \partial_\nu N_\mu)^2 + \frac{1}{2}(M-\frac{1}{2}\partial_\mu N_\mu)^2\ ] \quad , \qquad (36)
\end{aligned}
$$

where $\Delta\mathcal{L}_{tH}$ is 'tHooft's 1-loop counter-term formula [9] at the present approximation. This anomaly formula should be compared with the counter-term formula in the following points: 1) The total derivative terms have meaning in the anomaly formula; 2) Because the present approximation is weak-field expansion (of the 1-loop part) up to G_2, the cubic and quartic terms with respect to the external fields (N_μ, M) do not appear and will appear in G_3 and G_4 ; 3) The linear terms appear as total derivatives; 4) Symmetries, with respect to interchange of suffixes, are not assumed for the external fields N_μ and M; 5) The counter-term formula is obtained using the dimensional regularization, whereas the present anomaly formula is

obtained using the heat-kernel regularization. Although conformal anomalies were discussed in connection with the 1-loop counter-term formula[22], such a direct relation as above has not been known so far. The above formula (35) will be used in Sect.7.1

For the anomaly calculation of gravitational theories, we must consider the general case of $W_{\mu\nu}$. From the dimensional counting ($[W_{\mu\nu}]=(\mathrm{Mass})^0$, $[N_\mu]=(\mathrm{Mass})^1$, $[M]=(\mathrm{Mass})^2$, $[\partial_\mu]=(\mathrm{Mass})^1$), we see the t^0-part of $G_2(x,x;t)$ has three types of terms: 1) $W\times(\partial\partial\partial W,\ \partial\partial\partial N,\ \partial\partial M)$; 2) $(\partial W,N)\times(\partial\partial\partial W,\partial\partial N,\partial M)$; 3) $(\partial\partial W,\partial N,M)\times(\partial\partial W,\partial N,M)$. Among them, the most useful ones are type 3) terms, i.e. those terms which are composed only of $(\mathrm{Mass})^2$-dimensional quantities: $(\partial\partial W,\partial N,M)$,because they are sufficient to determine all anomaly terms[18]. They are given by

$$
\begin{aligned}
G_2(x,x;t)|_{t^0}^{(\partial\partial W,\partial N,M)} &= \frac{1}{(4\pi)^6}\int d^4v d^4u \int_0^1 dk \int_0^k dl \frac{1}{\{(1-k)(k-l)l\}^2} \\
\times e^{-\frac{1}{4}(\frac{v^2}{1-k}+\frac{(v-u)^2}{k-l}+\frac{u^2}{l})}\{ &\ \frac{1}{2}\partial_\alpha\partial_\beta W_{\mu\nu}(x)\cdot v^\alpha v^\beta(-\frac{\delta_{\mu\nu}}{2(k-l)}+\frac{(v-u)^\mu(v-u)^\nu}{4(k-l)^2}) \\
&+\partial_\alpha N_\mu(x)\cdot v^\alpha(-\frac{(v-u)^\mu}{2(k-l)})+M(x)\ \} \qquad (37) \\
\times\{\ \frac{1}{2}\partial_\gamma\partial_\delta W_{\lambda\sigma}(x)\cdot u^\gamma u^\delta(-\frac{\delta_{\lambda\sigma}}{2l}+\frac{u^\lambda u^\sigma}{4l^2}) &+\partial_\gamma N_\lambda(x)\cdot u^\gamma(-\frac{u^\lambda}{2l})+M(x)\ \} \quad .
\end{aligned}
$$

Other types, 1) and 2), are also similarly evaluated, but practically they are not necessary for the anomaly calculation.

4.3 Graphical Representation of Anomaly Formula

Further evaluation of (37) is straightforward, but we need to treat many terms. Here we introduce a graphical method to express those terms. Recently it has been shown that the graphical representation is practically useful to treat invariants and covariants in general relativity[23]. Because the connectivity of suffixes is visually expressed, it is very easy to discriminate between independent terms and dependent ones. Here we apply the technique to the present case: to represent global SO(n) ($n=4$ in the present case) covariants and invariants. We define the following graphical representation for $\partial_\alpha\partial_\beta W_{\mu\nu}$, $\partial_\alpha N_\mu$.

Fig.1

All independent terms which could appear in (37) are graphically listed up in App.C. They are those terms which satisfy the following conditions: 1) Invariants with respect to the global $SO(n)$ ($n=4$ in this section) trnasformation of the coordinate; 2) Dimension of (Mass)^4; 3) They are composed only of $\partial\partial W$, ∂N and M . Totally 26 terms appear. The final evaluation of (37) is given by

$$G_2(x,x;t)|_{t^0}^{(\partial\partial W,\partial N,M)} = \frac{1}{(4\pi)^2}\{\frac{1}{45}\partial_\sigma\partial_\lambda W_{\mu\nu}\cdot\partial_\sigma\partial_\nu W_{\mu\lambda} + \cdots \text{(see Table 1)} \quad \} \quad , (38)$$

and by Table 1 where the coefficients for all independent terms ,except the overall factor $1/(4\pi)^2$, are listed.

Graph	Expression	Coeff.	Graph	Expression	Coeff.
$\bar{A}1$	$\partial_\sigma\partial_\lambda W_{\mu\nu}\cdot\partial_\sigma\partial_\nu W_{\mu\lambda}$	1/45	$E1$	$\partial_\mu\partial_\lambda W_{\lambda\nu}\cdot\partial_\mu N_\nu$	1/12
$\bar{A}2$	$\partial_\sigma\partial_\lambda W_{\lambda\mu}\cdot\partial_\sigma\partial_\nu W_{\mu\nu}$	-2/45	$E2$	$\partial_\mu\partial_\lambda W_{\lambda\nu}\cdot\partial_\nu N_\mu$	1/12
$\bar{A}3$	$\partial_\sigma\partial_\lambda W_{\lambda\mu}\cdot\partial_\mu\partial_\nu W_{\nu\sigma}$	-2/45	$E3$	$\partial_\mu\partial_\nu W_{\lambda\lambda}\cdot\partial_\nu N_\mu$	0
$\bar{B}1$	$\partial_\nu\partial_\lambda W_{\sigma\sigma}\cdot\partial_\lambda\partial_\mu W_{\mu\nu}$	-1/90	$E4$	$\partial^2 W_{\mu\nu}\cdot\partial_\nu N_\mu$	0
$\bar{B}2$	$\partial^2 W_{\lambda\nu}\cdot\partial_\lambda\partial_\mu W_{\mu\nu}$	1/180	$\bar{Q}R$	$\partial_\mu\partial_\nu W_{\mu\nu}\cdot\partial_\lambda N_\lambda$	-1/6
$\bar{B}3$	$\partial_\mu\partial_\nu W_{\lambda\sigma}\cdot\partial_\mu\partial_\nu W_{\lambda\sigma}$	1/180	$\bar{P}R$	$\partial^2 W_{\mu\mu}\cdot\partial_\nu N_\nu$	1/24
$\bar{B}4$	$\partial_\mu\partial_\nu W_{\lambda\sigma}\cdot\partial_\lambda\partial_\sigma W_{\mu\nu}$	1/180	$F1$	$\partial_\mu N_\nu\cdot\partial_\mu N_\nu$	-1/24
$\bar{Q}^2$	$(\partial_\mu\partial_\nu W_{\mu\nu})^2$	1/18	$F2$	$\partial_\mu N_\nu\cdot\partial_\nu N_\mu$	-1/24
$\bar{C}1$	$\partial_\mu\partial_\nu W_{\lambda\lambda}\cdot\partial_\mu\partial_\nu W_{\sigma\sigma}$	1/360	RR	$(\partial_\mu N_\mu)^2$	1/8
$\bar{C}2$	$\partial^2 W_{\mu\nu}\cdot\partial^2 W_{\mu\nu}$	1/144	$M\bar{P}$	$M\cdot\partial^2 W_{\mu\mu}$	-1/12
$\bar{C}3$	$\partial_\mu\partial_\nu W_{\lambda\lambda}\cdot\partial^2 W_{\mu\nu}$	-1/90	$M\bar{Q}$	$M\cdot\partial_\mu\partial_\nu W_{\mu\nu}$	1/3
$\bar{P}\bar{Q}$	$\partial^2 W_{\lambda\lambda}\cdot\partial_\mu\partial_\nu W_{\mu\nu}$	-1/36	MR	$M\cdot\partial_\mu N_\mu$	-1/2
$\bar{P}^2$	$(\partial^2 W_{\lambda\lambda})^2$	1/288	MM	$M\cdot M$	1/2

Table 1 Anomaly Formula for $(\partial\partial W,\partial N,M)^2$-part of $G_2(x,x;t)|_{t^0}$
The overall factor is $1/(4\pi)^2$. Graph names are defined in App.C.

The result (33) for G_1 and the result of Table 1 for G_2 constitute the anomaly formula in 4 dim.

4.4 Weyl Anomaly in 4 Dimension

We apply the formula (33) and Table 1 to the Weyl anomaly calculation of the present example (1) with $n=4$. Here we introduce another graphical representation for the terms appearing in the weak-gravity expansion: $g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}$. We represent

$\partial_\mu \partial_\nu h_{\alpha\beta}$ as follows.

Fig.2

Then we can obtain the following graphical relations from (16).

(39)

We focus on $(\partial\partial h)^2$-terms in the anomaly because this type terms come only from $G_1(x,x;t)|_{t^0}$ and $(\partial\partial W, \partial N, M)$-part of $G_2(x,x;t)|_{t^0}$. All possible terms that could appear in the Weyl anomaly, are graphically listed up in App.B. Totally 13 independent terms appear as listed in Table 2. Inserting the above expressions(39) into $(\partial\partial W, \partial N, M)^2$-part of $G_2(x,x;t)|_{t^0}$(Table 1) we obtain G_2-Coeff. of Table 2, where the overall factor $1/(4\pi)^2$ is omitted. Inserting h^2-part of $(W_{\mu\nu}, N_\lambda, M)$, defined in (16), into $G_1(x,x;t)|_{t^0}$(33) and picking up $(\partial\partial h)^2$-terms, we obtain G_1-Coeff. of Table 2.

Graph	Expression	G_2-Coeff.	G_1-Coeff.	G_1+G_2
$A1$	$\partial_\sigma\partial_\lambda h_{\mu\nu}\cdot\partial_\sigma\partial_\nu h_{\mu\lambda}$	$1/45$	$-7/180$	$-1/60$
$A2$	$\partial_\sigma\partial_\lambda h_{\lambda\mu}\cdot\partial_\sigma\partial_\nu h_{\mu\nu}$	$-1/45\cdot 8$	$-1/90$	$-1/72$
$A3$	$\partial_\sigma\partial_\lambda h_{\lambda\mu}\cdot\partial_\mu\partial_\nu h_{\nu\sigma}$	$-1/45\cdot 8$	0	$-1/360$
$B1$	$\partial_\nu\partial_\lambda h_{\sigma\sigma}\cdot\partial_\lambda\partial_\mu h_{\mu\nu}$	$-1/90$	$2/3\cdot 24$	$1/60$
$B2$	$\partial^2 h_{\lambda\nu}\cdot\partial_\lambda\partial_\mu h_{\mu\nu}$	$1/180$	$-4/15\cdot 24$	$-1/180$
$B3$	$\partial_\mu\partial_\nu h_{\lambda\sigma}\cdot\partial_\mu\partial_\nu h_{\lambda\sigma}$	$1/180$	$1/5\cdot 24$	$-1/72$
$B4$	$\partial_\mu\partial_\nu h_{\lambda\sigma}\cdot\partial_\lambda\partial_\sigma h_{\mu\nu}$	$1/180$	0	$1/180$
Q^2	$(\partial_\mu\partial_\nu h_{\mu\nu})^2$	0	0	0
$C1$	$\partial_\mu\partial_\nu h_{\lambda\lambda}\cdot\partial_\mu\partial_\nu h_{\sigma\sigma}$	$1/360$	$-1/144$	$-1/240$
$C2$	$\partial^2 h_{\mu\nu}\cdot\partial^2 h_{\mu\nu}$	$1/144$	$-1/15\cdot 24$	$1/240$
$C3$	$\partial_\mu\partial_\nu h_{\lambda\lambda}\cdot\partial^2 h_{\mu\nu}$	$-1/90$	$1/3\cdot 24$	$1/360$
PQ	$\partial^2 h_{\lambda\lambda}\cdot\partial_\mu\partial_\nu h_{\mu\nu}$	0	0	0
P^2	$(\partial^2 h_{\lambda\lambda})^2$	0	0	0

Table 2 Weyl Anomaly of 4 Dim Gravity-Scalar Theory: $(\partial\partial h)^2$-part
The overall factor is $1/(4\pi)^2$. Graph names are defined in App.B.

Now we have evaluated the $(\partial\partial h)^2$-part of the Weyl anomaly completely. They are expressed by the $(\partial\partial h)^2$-part of the following invariant quantities.

$$\text{Weyl Anomaly} \quad = \frac{1}{(4\pi)^2}\sqrt{g}(\alpha_1\nabla^2 R+\beta_1 R^2+\beta_2 R_{\mu\nu}R^{\mu\nu}+\beta_3 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}) \quad ,$$
$$\alpha_1=-\frac{1}{180} \quad , \quad \beta_1=0 \quad , \quad \beta_2=-\frac{1}{180} \quad , \quad \beta_3=\frac{1}{180} \quad .(40)$$

The above coefficients are obtained by the weak gravity expansion of the right-hand side and equating the result with that of Table 2. Weak expansion of all invariants of the right-hand side is provided in Table 4 of App.B. From the general coordinate transformation invariance, the result (40) is correct to all orders with respect to $h_{\mu\nu}$.

Here we comment on the reason why the present formulae are sufficient to determine the anomaly terms completely. In $n=4$ dim case, the number of independent (general coordinate) invariants which could appear in the Weyl anomaly is 4 as shown in (40). Whereas the number of independent (global SO(4)) invariants of $(\partial\partial h)^2$ is 13. Since each term of $(\partial\partial h)^2$ gives an independent constraint, $(\partial\partial h)^2$-terms are sufficient to detrmine the Weyl anomaly. Generally, in n-dim, the number of independent terms which appear in the Weyl anomaly is far less than that of $(\partial\partial h)^{n/2}$-terms. Hence the formulae up to $G_{n/2}$ is sufficient. Furthermore we need not all terms contained in $G_i \ : \ i=1,2,\cdots,\frac{n}{2}$. In $n=4$ dim case, we have obtained only $(\partial\partial W,\partial N,M)^2$-type of G_2. Other types do not contribute to $(\partial\partial h)^2$-terms. This is the same situation in the general n-dim case. Although we expalin for the case of the Weyl anomaly, it is valid for other anomalies.

5 Weyl Anomaly in Fermion-Gravity System

In the fermion-gravity system, we must treat the vielbein field instead of the metric field. We explain the weak field expansion and the graphical representation in this case, by taking an example of the Weyl anomaly calculation for the system of the Dirac field coupled to gravity in n dimension. The Lagrangian is given by

$$\mathcal{L}[g_{\mu\nu},\psi]=\frac{1}{2}\sqrt{g}\bar{\psi}i\overleftrightarrow{\nabla\!\!\!\!/}\,\psi, \tag{41}$$

where the notations and conventions are defined by Appendix A. This Lagrangian is invariant under the following local Weyl transformation:

$$\begin{gathered}g^{\mu\nu}(x)'=e^{2\alpha(x)}g^{\mu\nu}(x)\quad,\\ \psi(x)'=\exp\{\tfrac{n-1}{2}\alpha(x)\}\ \psi(x)\quad,\quad\bar{\psi}(x)'=\exp\{\tfrac{n-1}{2}\alpha(x)\}\ \bar{\psi}(x)\quad,\end{gathered} \tag{42}$$

where the notation is the same as Eq. (2).

We redefine the fields as

$$\tilde{\psi}\equiv g^{1/4}\psi,\qquad\tilde{\bar{\psi}}\equiv g^{1/4}\bar{\psi}, \tag{43}$$

for the path integral measures to be the general coordinate invariant[10]. In terms of the redefined fields, the Weyl transformation is given by

$$\tilde{\psi}(x)'=e^{-\alpha(x)/2}\tilde{\psi}(x)\quad,\quad\tilde{\bar{\psi}}(x)'=e^{-\alpha(x)/2}\tilde{\bar{\psi}}(x). \tag{44}$$

Then the Jacobian of the above Weyl transformation is written as

$$\begin{aligned}J_{Dirac}&\equiv\det\frac{(\partial\tilde{\psi}(y),\partial\tilde{\bar{\psi}}(y))}{(\partial\tilde{\psi}'(x),\partial\tilde{\bar{\psi}}'(x))}=\det\ (e^{\alpha(x)}\delta^n(x-y)\ )\\&=\exp(\mathrm{Tr}\ [\alpha(x)\delta^n(x-y)]+O(\alpha^2))\quad.\end{aligned} \tag{45}$$

We regularize the delta function $\delta^n(x-y)$ as in the scalar field case.

$$\begin{aligned}G(x,y;t)\equiv<x|e^{-t\vec{D}}|y>\quad,\quad t>0\quad,\\ \vec{D}_x\equiv-(\sqrt[4]{g}\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\frac{1}{\sqrt[4]{g}})_x.\end{aligned} \tag{46}$$

Then we can express the Weyl anomaly as

$$\ln J_{Dirac}=\lim_{t\to+0}\mathrm{Tr}[\alpha(x)G(x,y;t)]+O(\alpha^2)\ . \tag{47}$$

Note that the trace contains contraction of the fermion indices.

Since the action is described by the vielbein, we take the following weak-field expansion of the vielbein:

$$e_\mu{}^a=\delta_\mu{}^a+f_\mu{}^a\quad, \tag{48}$$

to solve the differential equation for $G(x,y;t)$ perturbatively. The spin connection and the curvature tensor is expanded as

$$
\begin{aligned}
\omega_\mu{}^{ab} &= \frac{1}{2}(-\partial_\mu f_\nu{}^a \delta_\nu{}^b + \partial_\nu f_\mu{}^a \delta_\nu{}^b + \partial_\nu f_{\lambda c}\delta_{\mu c}\delta_\lambda{}^a \delta_\nu{}^b - (a \leftrightarrow b)) + O(f^2), \\
R^\lambda{}_{\rho\mu\nu} &= \frac{1}{2}\partial_\mu\partial_\rho f_{\lambda a}\delta_{\nu a} + \frac{1}{2}\partial_\mu\partial_\rho f_{\nu a}\delta_{\lambda a} - \frac{1}{2}\partial_\mu\partial_\lambda f_{\nu a}\delta_{\rho a} - \frac{1}{2}\partial_\mu\partial_\lambda f_{\rho a}\delta_{\nu a} \\
&\quad -(\mu \leftrightarrow \nu) + O(f^2).
\end{aligned}
\tag{49}
$$

From the above expression, we obtain $W_{\mu\nu}$, N_μ and M ,in the expanded form, as

$$
\begin{aligned}
W_{\mu\nu} &= g^{\mu\nu} - \delta^{\mu\nu} = -f_{\mu a}\delta_{\nu a} - f_{\nu a}\delta_{\mu a} + O(f^2), \\
N_\lambda &= -g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} + 2g^{\mu\lambda}\omega_\mu - g^{\mu\lambda}\Gamma^\nu_{\mu\nu} \\
&= -(\partial_\mu f_{\mu a}\delta_{\lambda a} + \partial_\mu f_{\lambda a}\delta_{\mu a}) + \omega_\lambda{}^{ab(1)}\overset{\circ}{\sigma}{}^{ab} + O(f^2), \\
M &= -g^{\mu\nu}(\Gamma^\lambda{}_{\mu\nu}\omega_\lambda + \Gamma^\lambda_{\mu\lambda}\omega_\nu) + g^{\mu\nu}(\omega_\mu\omega_\nu + \partial_\mu\omega_\nu) + \frac{1}{2}\sigma^{\mu\nu}\sigma^{\lambda\rho}R_{\mu\nu\lambda\rho} \\
&\quad + g^{\mu\nu}(-\frac{1}{2}\partial_\mu\Gamma^\lambda_{\nu\lambda} + \frac{1}{4}\Gamma^\lambda_{\mu\lambda}\Gamma^\sigma_{\nu\sigma} + \frac{1}{2}\Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma}) \\
&= \delta_\mu{}^c\delta_\nu{}^d\overset{\circ}{\sigma}{}^{cd}\overset{\circ}{\sigma}{}^{ab}\partial_\mu\omega_\nu{}^{ab(1)} + \frac{1}{2}\partial_\mu\omega_\mu{}^{ab(1)}\overset{\circ}{\sigma}{}^{ab} - \frac{1}{2}\partial^2 f_\mu{}^a\delta_{\mu a} + O(f^2),
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
\sigma^{\mu\nu} &= e^\mu{}_a e^\nu{}_b \overset{\circ}{\sigma}{}^{ab}, \\
\omega_\mu{}^{ab(1)} &= \frac{1}{2}(-\partial_\mu f_\nu{}^a\delta_\nu{}^b + \partial_\nu f_\mu{}^a\delta_\nu{}^b + \partial_\nu f_{\lambda c}\delta_{\mu c}\delta_\lambda{}^a\delta_\nu{}^b - (a \leftrightarrow b)).
\end{aligned}
\tag{51}
$$

Here we introduce the graphical representation for $\partial_\mu\partial_\nu f_\alpha{}^a$ and $\delta_{\mu a}$ as in Fig.3a and 3b respectively.

Fig.3

From the relation $h_{\mu\nu} = f_\mu{}^a\delta_{\nu a} + (\mu \leftrightarrow \nu) + O(f^2)$, we obtain a relation between Fig.2 and Fig.3a, as shown in Fig.4a. In Fig.4b the 'anti-symmetrized' partner of

Fig.4a (we call it 'anti-symmetric metric ') is newly defined.

Fig.4

Graphically $\partial\partial W$, ∂N and M are expressed as

Fig.5

In Fig.5, I is the $n \times n$ unit matrix.

In order to obtain the Weyl anomaly in 4 dim, we substitute the above expressions for $n = 4$ to the 4 dim anomaly formula of $G_1(x,x;t)|_{t^0}$ and $G_2(x,x;t)|_{t^0}$. We notice the anti-symmetric metric looks to appear generally. In the torsion-less theory of the present example, we know those terms cancel each other and disappear. For purely practical reason, we can take the following gauge for the external gravitational filed.

$$f_\mu{}^a \delta_{\nu a} = f_\nu{}^a \delta_{\mu a}$$

graphical rep. of gauge (52)

Now the perturbation is done by the power of $f_{\mu\nu} \equiv f_\mu{}^a \delta_{\nu a} + f_\nu{}^a \delta_{\mu a}$ which is related with $h_{\mu\nu}$ as

$$h_{\mu\nu} = f_{\mu\nu} + f_\mu{}^a f_{\nu a} \quad , \quad f_{\mu\nu} \equiv f_\mu{}^a \delta_{\nu a} + f_\nu{}^a \delta_{\mu a} \quad . \tag{53}$$

$(\partial\partial f)^2$-terms are obtained by substituting f-part of $(W_{\mu\nu}, N_\mu, M)$, (50), into $G_2(x,x;t)|_{t^0}$ and by substituting f^2-part of $(W_{\mu\nu}, N_\mu, M)$ into $G_1(x,x;t)|_{t^0}$. Each

contribution is separately listed in Table 3. Since $f_{\mu\nu}$ has the same symmetry as $h_{\mu\nu}$,with respect to the suffix-exchange, all possible terms are the same as $(\partial\partial h)^2$.

Graph	Expression	G_2-Coeff.	G_1-Coeff.	G_1+G_2
$A1'$	$\partial_\sigma\partial_\lambda f_{\mu\nu}\cdot\partial_\sigma\partial_\nu f_{\mu\lambda}$	$4/45$	$-1/30$	$1/18$
$A2'$	$\partial_\sigma\partial_\lambda f_{\lambda\mu}\cdot\partial_\sigma\partial_\nu f_{\mu\nu}$	$-1/90$	$1/20$	$7/180$
$A3'$	$\partial_\sigma\partial_\lambda f_{\lambda\mu}\cdot\partial_\mu\partial_\nu f_{\nu\sigma}$	$-1/90$	0	$-1/90$
$B1'$	$\partial_\nu\partial_\lambda f_{\sigma\sigma}\cdot\partial_\lambda\partial_\mu f_{\mu\nu}$	$-2/45$	0	$-2/45$
$B2'$	$\partial^2 f_{\lambda\nu}\cdot\partial_\lambda\partial_\mu f_{\mu\nu}$	$1/45$	$1/20$	$13/180$
$B3'$	$\partial_\mu\partial_\nu f_{\lambda\sigma}\cdot\partial_\mu\partial_\nu f_{\lambda\sigma}$	$23/360$	$-1/10$	$-13/360$
$B4'$	$\partial_\mu\partial_\nu f_{\lambda\sigma}\cdot\partial_\lambda\partial_\sigma f_{\mu\nu}$	$-7/360$	0	$-7/360$
$Q^{2'}$	$(\partial_\mu\partial_\nu f_{\mu\nu})^2$	$1/72$	0	$1/72$
$C1'$	$\partial_\mu\partial_\nu f_{\lambda\lambda}\cdot\partial_\mu\partial_\nu f_{\sigma\sigma}$	$1/90$	0	$1/90$
$C2'$	$\partial^2 f_{\mu\nu}\cdot\partial^2 f_{\mu\nu}$	$1/36$	$-1/20$	$-1/45$
$C3'$	$\partial_\mu\partial_\nu f_{\lambda\lambda}\cdot\partial^2 f_{\mu\nu}$	$-2/45$	0	$-2/45$
PQ'	$\partial^2 f_{\lambda\lambda}\cdot\partial_\mu\partial_\nu f_{\mu\nu}$	$-1/36$	0	$-1/36$
$P^{2'}$	$(\partial^2 f_{\lambda\lambda})^2$	$1/72$	0	$1/72$

Table 3 Weyl Anomaly of 4 Dim Gravity-Fermion Theory: $(\partial\partial f)^2$-part
The overall factor is $1/(4\pi)^2$. Graph names are defined in App.B.

In order to express the above result covariantly, we need $(\partial\partial f)^2$-part of the 4 invariants. As for R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, the expansion coefficients are the same as those given in Table 4. As for $\nabla^2 R$, however, the difference between the $f_{\mu\nu}$-perturbation and $h_{\mu\nu}$-perturbation appears due to the existence of the linear term of $h_{\mu\nu}$ in $\nabla^2 R$.

$$\nabla^2 R=\partial^2\partial^2 h-\partial^2\partial_\alpha\partial_\beta h_{\alpha\beta}+O(h^2)\quad,\quad g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}\quad. \tag{54}$$

$O(h^2)$-part gives the same expansion coefficients as given in Table 4. The $h_{\mu\nu}$-linear terms give additional $(\partial\partial f)^2$-terms due to the f^2-term in (53). So the second column of Table 4, for the $(\partial\partial f)^2$-coefficients, should be replaced by

$$\nabla^2 R|_{(\partial\partial f)^2}=(A1')+2(A2')-2(B1')+2(B2')-\frac{3}{2}(B3')$$
$$+\frac{1}{2}(C1')-(C2')-(C3')\ +\{\ -\frac{1}{2}(A1')-\frac{1}{2}(A2')-\frac{1}{2}(B2')+(B3')+\frac{1}{2}(C2')\ \}\ , \tag{55}$$

where the bracket part ($\{\cdots\}$) comes from the $h_{\mu\nu}$-linear terms. Finally the above result of Table 3 can be written by the following invariant quantities:

$$\begin{aligned}\text{Weyl Anomaly}\quad &\equiv\quad \frac{\delta}{\delta\alpha(x)}\ \ln\ J_{Dirac}|_{\alpha=0}\\ &=\quad \sqrt{g}(\alpha_1\nabla^2R+\beta_1R^2+\beta_2R_{\mu\nu}R^{\mu\nu}+\beta_3R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma})\quad ,\end{aligned} \tag{56}$$

with the following coefficients:

$$\alpha_1=\frac{1}{30}\quad,\quad\beta_1=\frac{1}{72}\quad,\quad\beta_2=-\frac{1}{45}\quad,\quad\beta_3=-\frac{7}{360}. \tag{57}$$

This is the same result as the other calculation of the Weyl anomaly of four-dimensional Dirac field[24].

6 Weyl, Gravitational and Local Lorentz Anomaly in 2 Dim Fermion-Gravity System

$4k+2$ dimensional Weyl fermion coupled to gravity can be anomalous with respect to the local Lorentz and general coordinate transformation[6]. The Lagrangian is expressed by

$$\mathcal{L}[g_{\mu\nu},\psi] = \sqrt{g}\frac{1}{2}\bar{\psi}i\overleftrightarrow{\nabla\!\!\!\!/}\left(\frac{1-\gamma_5}{2}\right)\psi. \tag{58}$$

The local Lorentz and general coordinate transformation of the fermion fields are following:

$$\begin{aligned} \delta_{LL}(\alpha_{ab})\tilde{\psi} &= -\frac{1}{2}\alpha_{ab}\mathring{\sigma}^{ab}\tilde{\psi}, \\ \delta_G(\xi^\mu)\tilde{\psi} &= -\frac{1}{2}\tilde{\psi}\nabla_\mu\xi^\mu - \xi^\mu\tilde{\nabla}_\mu\tilde{\psi} + \xi^\mu\omega_\mu\tilde{\psi},, \\ \tilde{\nabla}_\mu &= \sqrt[4]{g}\nabla_\mu\frac{1}{\sqrt[4]{g}} = \sqrt[4]{g}(\partial_\mu+\omega_\mu)\frac{1}{\sqrt[4]{g}} \quad , \end{aligned} \tag{59}$$

where $\tilde{\psi}$ is defined in Eq.(43), and α_{ab} is a local Lorentz transformation gauge parameter, ξ^μ is a general coordinate transformation parameter and $\omega_\mu = \frac{1}{2}\mathring{\sigma}^{ab}\omega_{\mu ab}$. We take the following transformation instead of δ_G.

$$\delta_{cov}\tilde{\psi} = [\delta_G(\xi^\mu) + \delta_{LL}(\xi^\mu\omega_{\mu ab})]\tilde{\psi} = -\frac{1}{2}\tilde{\psi}\nabla_\mu\xi^\mu - \xi^\mu\tilde{\nabla}_\mu\tilde{\psi} \quad . \tag{60}$$

We define the following operators:

$$D\!\!\!\!/_L \equiv \sqrt[4]{g}\nabla\!\!\!\!/\frac{1}{\sqrt[4]{g}}\left(\frac{1-\gamma_5}{2}\right), \qquad D\!\!\!\!/_L{}^\dagger \equiv \sqrt[4]{g}\nabla\!\!\!\!/\frac{1}{\sqrt[4]{g}}\left(\frac{1+\gamma_5}{2}\right), \tag{61}$$

Then the Lagrangian is written as

$$\mathcal{L}[g_{\mu\nu},\psi] = \frac{1}{2}\bar{\tilde{\psi}}i\overleftrightarrow{D\!\!\!\!/}_L\tilde{\psi}. \tag{62}$$

In this section, we explicitly calculate 2 dim local Lorentz and general coordinate anomaly using the anomaly formulae.

6.1 Local Lorentz Anomaly in 2 Dim Weyl Fermion

We write the Jacobian of local Lorentz transformation of measure $\mathcal{D}\tilde{\bar{\psi}}\mathcal{D}\tilde{\psi}$ as J_{LL}. The Jacobian is written as

$$\begin{aligned}\ln J_{LL} &= (\ln J_{LL})_{\tilde{\psi}} + (\ln J_{LL})_{\tilde{\bar{\psi}}} \\ &= \frac{1}{2}\mathrm{Tr}_{\tilde{\psi}}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab}\delta^2(x-y)\right] - \frac{1}{2}\mathrm{Tr}_{\tilde{\bar{\psi}}}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab}\delta^2(x-y)\right].\end{aligned} \tag{63}$$

Because $\not{D}_L$ is not hermitian, we have some freedom in the choice of $\vec{D}_x$[5]. We take it in such a way that the covariant anomaly is obtained. So we regularize the delta functions in (63) as

$$\begin{aligned}\ln J_{LL} &= \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab} < x|\exp(t\not{D}_L{}^{\dagger}\not{D}_L)|y>\right] \\ &\quad - \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab} < x|\exp(t\not{D}_L\not{D}_L{}^{\dagger})|y>\right].\end{aligned} \tag{64}$$

(64) is rewritten as

$$\begin{aligned}\ln J_{LL} &= \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab} < x|\left(\frac{1+\gamma_5}{2}\right)\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|y>\right] \\ &\quad - \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab} < x|\left(\frac{1-\gamma_5}{2}\right)\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|y>\right] \\ &= \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\alpha_{ab}(x)\overset{\circ}{\sigma}_{ab}\gamma_5 < x|\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|y>\right].\end{aligned} \tag{65}$$

$$\begin{aligned}G(x,y;t) = &< x|\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|y> \quad , \\ &\vec{D}_x = -(\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})_x,\end{aligned} \tag{66}$$

which is the same as n dim Dirac fermion case (46). The expressions for $W_{\mu\nu}(x)$, $N_\mu(x)$ and $M(x)$ are (50).

To calculate $G(x,x;t) =< x|\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|x>$, we use the 2 dim formula (27) in Sec.3 and substitute (50) to (27). Then we obtain

$$\begin{aligned}G_1(x,x;t) &= \frac{1}{4\pi}\Big[\frac{1}{6}\partial^2(f_{\mu a}\delta_{\mu a}) - \frac{2}{3}\partial_\mu\partial_\nu(f_{\mu a}\delta_{\nu a}) + \frac{1}{2}\partial_\lambda(2\partial_\mu f_{\mu a}\delta_{\lambda a} - \partial_\lambda f_{\mu a}\delta_{\mu a}) \\ &\quad + \delta_{\mu c}\delta_{\nu d}\overset{\circ}{\sigma}{}^{cd}\overset{\circ}{\sigma}{}^{ab}\partial_\mu\omega_{\nu ab}{}^{(1)}\Big] + O(f^2),\end{aligned} \tag{67}$$

where $\omega_{\nu ab}{}^{(1)}$ is defined by Eq. (51). Since $\overset{\circ}{\sigma}{}^{ab} = \frac{i}{2}\gamma_5\epsilon^{ab}$ is valid in 2 dim and the trace of the odd number products of γ_5 vanishes, the anomaly term is calculated as

$$\begin{aligned}A_{LL} &\equiv \alpha_{ab}(x)\frac{\partial}{\partial\alpha_{ab}(x)}\ \ln\ J_{LL} \\ &= \frac{1}{4\pi}\alpha_{ab}\epsilon^{ab}\frac{i}{12}\left[\frac{1}{3}\partial^2 f_{\mu a}\delta_{\mu a} - \frac{1}{3}\partial_\mu\partial_\nu f_{\mu a}\delta_{\nu a}\right] + O(f^2) \\ &= \alpha_{ab}\frac{1}{4\pi}\frac{i}{24}\epsilon^{ab}\sqrt{g}R \quad .\end{aligned} \tag{68}$$

This reproduces the result of [6, 25].

6.2 General Coordinate Anomaly in 2 Dim Weyl Fermion

The calculation of the general coordinate anomaly is slightly technical. Let us calculate the gravitational anomaly for the transformation δ_{cov}. Compared with all other transformations in the present text, only the general coordinate one has the (covariant) derivative term. The Jacobian of the measure $\mathcal{D}\tilde{\bar{\psi}}\mathcal{D}\tilde{\psi}$ with respect to the transformation δ_{cov} denoted as J_{cov} is written as

$$
\begin{aligned}
\ln J_{cov} &= (\ln J_{cov})_{\tilde{\psi}} + (\ln J_{cov})_{\tilde{\bar{\psi}}} \\
&= \frac{1}{2}\mathrm{Tr}_{\tilde{\psi}}\left[\left(\frac{1}{2}\nabla_\mu \xi^\mu(x) + \xi^\mu(x)\tilde{\nabla}_\mu\right)\delta^2(x-y)\right] \\
&\quad + \frac{1}{2}\mathrm{Tr}_{\tilde{\bar{\psi}}}\left[\left(\frac{1}{2}\nabla_\mu \xi^\mu(x) + \overleftarrow{\tilde{\nabla}}_\mu \xi^\mu(x)\right)\delta^2(x-y)\right].
\end{aligned}
\tag{69}
$$

We can regularize the Jacobian as

$$
\begin{aligned}
\ln J_{cov} &= \lim_{t\to+0}\mathrm{Tr}\Big[< x|\left(\frac{1}{2}\nabla_\mu \xi^\mu(x) + \xi^\mu(x)\tilde{\nabla}_\mu\right)\exp(t\not{D}_L{}^\dagger \not{D}_L)|y> \\
&\quad + < x|\left(\frac{1}{2}\nabla_\mu \xi^\mu(x) + \overleftarrow{\tilde{\nabla}}_\mu \xi^\mu(x)\right)\exp(t\not{D}_L \not{D}_L{}^\dagger)|y>\Big] \\
&= \lim_{t\to+0}\mathrm{Tr}\Big[< x|\left(\frac{1}{2}\nabla_\mu \xi^\mu(x) + \xi^\mu(x)\tilde{\nabla}_\mu\right)\left(\frac{1+\gamma_5}{2}\right)\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})_y|y> \\
&\quad + < x|\left(\frac{1}{2}\nabla_\mu \xi^\mu(x) + \overleftarrow{\tilde{\nabla}}_\mu \xi^\mu(x)\right)\left(\frac{1-\gamma_5}{2}\right)\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|y>\Big] \\
&= \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\gamma_5 < x|\sqrt[4]{g}(\xi^\mu(x)\nabla_\mu + \nabla_\mu\xi^\mu(x))\frac{1}{\sqrt[4]{g}}\exp(t\sqrt[4]{g}\not{\nabla}\not{\nabla}\frac{1}{\sqrt[4]{g}})|y>\right] \\
&= \lim_{t\to+0}\left(\frac{1}{2}\right)\mathrm{Tr}\left[\frac{1}{t}\gamma_5 < x|\exp\{t\sqrt[4]{g}(\not{\nabla}\not{\nabla} + \xi^\mu\nabla_\mu + \nabla_\mu\xi^\mu)\frac{1}{\sqrt[4]{g}}\}|y>\right]\Bigg|_{\xi^1}.
\end{aligned}
\tag{70}
$$

In the last expression, taking the linear term of ξ is understood. This trick was taken in [5]. So we consider the following heat-kernel.

$$
\begin{aligned}
G(x,y;t) &=< x|\exp\{t\sqrt[4]{g}(\not{\nabla}\not{\nabla} + \xi^\mu\nabla_\mu + \nabla_\mu\xi^\mu)\frac{1}{\sqrt[4]{g}}\}|y>, \\
\vec{D}_x &= -\left(\sqrt[4]{g}(\not{\nabla}\not{\nabla} + \xi^\mu\nabla_\mu + \nabla_\mu\xi^\mu)\frac{1}{\sqrt[4]{g}}\right)_x.
\end{aligned}
\tag{71}
$$

Then $W_{\mu\nu}$, N_μ and M , corresponding to (71), is given by, up to the linear order of $f_{\mu a}$,

$$
\begin{aligned}
W_{\mu\nu} &= g^{\mu\nu} - \delta^{\mu\nu} = -f^{\mu a}\delta^{\nu a} - f^{\nu a}\delta^{\mu a} + O(f^2), \\
N^\lambda &= -g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} - g^{\mu\lambda}\Gamma^\nu{}_{\mu\nu} + 2g^{\mu\lambda}\omega_\mu + 2\xi^\lambda \\
&= -(\partial_\mu f_{\mu a}\delta_{\lambda a} + \partial_\mu f_{\lambda a}\delta_{\mu a}) + \omega_\lambda{}^{ab(1)}\overset{\circ}{\sigma}{}^{ab} + 2\xi_\lambda + O(f^2), \\
M &= -g^{\mu\nu}(\Gamma^\lambda{}_{\mu\nu}\omega_\lambda + \Gamma^\lambda{}_{\mu\lambda}\omega_\nu) + g^{\mu\nu}(-\frac{1}{2}\partial_\mu\Gamma^\lambda_{\nu\lambda} + \frac{1}{4}\Gamma^\lambda_{\mu\lambda}\Gamma^\sigma_{\nu\sigma} + \frac{1}{2}\Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma}) \\
&\quad + \frac{1}{2}\sigma^{\mu\nu}\sigma^{\lambda\rho}R_{\mu\nu\lambda\rho} + g^{\mu\nu}\omega_\mu\omega_\nu + g^{\mu\nu}\partial_\mu\omega_\nu + 2\xi^\mu\omega_\mu + \partial_\mu\xi^\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \delta_\mu{}^c\delta_\nu{}^d\overset{\circ}{\sigma}{}^{cd}\overset{\circ}{\sigma}{}^{ab}\partial_\mu\omega_\nu{}^{ab(1)} + \frac{1}{2}\partial_\mu\omega_\mu{}^{ab(1)}\overset{\circ}{\sigma}{}^{ab} - \frac{1}{2}\partial^2 f_\mu{}^a\delta_{\mu a} + \xi_\mu\omega_\mu{}^{ab(1)}\overset{\circ}{\sigma}{}^{ab} + \partial_\mu\xi_\mu \\
&\quad -\xi^\mu\partial_\mu f_\nu{}^a\delta_{a\nu} + O(f^2).
\end{aligned} \tag{72}
$$

Since the last expression of (70) has the factor t^{-1}, and has the expansion variable ξ^μ in addition to $f_{\mu a}$, we need two dim formula of $G_1(x,y;t)|_{t^1}$ and $G_2(x,y;t)|_{t^1}$ to obtain the finite term at $t \to +0$.

From the formulae (24) and (25), we note that

$$
\frac{1}{4\pi t}G_l^{(4)}(x,y;t)|_{t^0} = G_l^{(2)}(x,y;t)|_{t^1}, \qquad \text{for} \quad l = 1, 2, \cdots, \tag{73}
$$

where $G_l^{(n)}(x,y;t)$ is $G_l(x,y;t)$ in n dimension. Therefore, from the result (33) in Sec.4, we obtain the two dimensional formula of $G_1(x,x;t)|_{t^1}$ as

$$
\begin{aligned}
G_1(x,x;t)|_{t^1} &= \frac{t}{4\pi}\Big[-\frac{1}{120}(\partial^2)^2 W_{\mu\mu}(x) + \frac{1}{20}\partial^2\partial_\mu\partial_\nu W_{\mu\nu}(x) \\
&\quad -\frac{1}{12}\partial^2\partial_\mu N_\mu(x) + \frac{1}{6}\partial^2 M(x)\Big],
\end{aligned} \tag{74}
$$

This term produces the only total divergence term, so it does not contribute to the gravitational anomaly. Next we calculate the contribution from $G_2(x,y;t)|_{t^1}$. Since the trace of the odd number products of γ_5 vanishes and we need the terms proportional to $(\xi_\mu)^1$, the necessary terms in $G_2(x,y;t)|_{t^1}$ are $W_{\mu\nu}$-independent terms. They are given by, from (35),

$$
\begin{aligned}
G_2(x,x;t)|_{t^1} &= \frac{t}{4\pi}\Big[-\frac{1}{12}(\partial^2 N_\mu(x))N_\mu(x) - \frac{1}{24}\partial_\mu N_\nu(x)\partial_\mu N_\nu(x) \\
&-\frac{1}{24}\partial_\nu N_\mu(x)\partial_\mu N_\nu(x) + \frac{1}{8}\partial_\mu N_\mu(x)\partial_\nu N_\nu(x) - \frac{1}{2}\partial_\mu N_\mu(x)M(x) + \frac{1}{2}M(x)M(x)\Big],
\end{aligned} \tag{75}
$$

Substituting the expression (72) to (75), we obtain

$$
G_2(x,x;t)|_{t^1} = \frac{t}{4\pi}\Big[-\frac{1}{6}\xi_\mu\partial^2\omega_\mu{}^{ab}\overset{\circ}{\sigma}{}^{ab} + \frac{1}{6}\xi_\mu\partial_\mu\partial_\nu\omega_\nu{}^{ab}\overset{\circ}{\sigma}{}^{ab}\Big] + \text{tot. div} + O(f^2), \tag{76}
$$

[tot. div] is the total divergence terms, which does not produce the gravitational anomaly. From (70) and (76), we obtain the following result:

$$
\begin{aligned}
A_{cov}(x) &\equiv \xi_\mu(x)\frac{\partial}{\partial\xi_\mu(x)}\ln\ J_{cov} \\
&= \frac{1}{4\pi}\frac{i}{12}\xi_\mu\epsilon_{\nu\lambda}[\partial^2\partial_\nu f_{\mu a}\delta_{\lambda a} + \partial^2\partial_\nu f_{\lambda a}\delta_{\mu a} \\
&\quad -\partial_\mu\partial_\rho\partial_\lambda f_{\rho a}\delta_{\nu a} - \partial_\mu\partial_\rho\partial_\nu f_{\lambda a}\delta_{\rho a}] + O(f^2) \\
&= \frac{1}{4\pi}\frac{i}{12}\xi_\mu\sqrt{g}e^{\nu\lambda}\nabla^\rho R^\mu{}_{\rho\nu\lambda} \\
&= -\frac{1}{4\pi}\frac{i}{12}\xi_\mu\sqrt{g}e^{\mu\nu}\nabla_\nu R,
\end{aligned} \tag{77}
$$

which reproduces the known result about the gravitational anomaly[6, 25].

The relation (73) suggests that two dimensional gravitational anomaly is related to the four dimensional certain anomaly. It is the chiral anomaly. Their relations will be closely investigated in Sec.7.2.

6.3 Relations of Anomalies

In this subsection and in 7.2, we examine relations among some anomalies. Let us consider the Weyl transformation of two-dimensional Dirac field. The Lagrangian is (41) and the Weyl transformation is the same as Eq. (42). So we obtain the Weyl anomaly of two dimensional Dirac field coupled to gravity as follows:

$$\begin{aligned} \ln J^{2d}_{Dirac} &= \lim_{t\to+0} \mathrm{Tr}[\alpha(x) < x|e^{-t\vec{D}}|y >] \quad . \\ G(x,y;t) &\equiv < x|e^{-t\vec{D}}|y > \quad , \quad t>0 \quad , \\ \vec{D}_x &\equiv -(\sqrt[4]{g}\not\nabla\not\nabla\frac{1}{\sqrt[4]{g}})_x \quad . \end{aligned} \tag{78}$$

Because $\overset{\circ}{\sigma}{}^{ab} = \frac{i}{2}\gamma_5\epsilon^{ab}$ is valid in 2 dim, the local Lorentz anomaly (65) becomes

$$\ln J_{LL} \quad = \quad \lim_{t\to+0}\left(\frac{i}{4}\right)\mathrm{Tr}\left[\alpha_{ab}(x)\epsilon_{ab} < x|\exp(t\sqrt[4]{g}\not\nabla\not\nabla\frac{1}{\sqrt[4]{g}})|y >\right] \quad . \tag{79}$$

By defining

$$K(x) \quad = \quad \lim_{t\to+0}\mathrm{tr}\left[< x|\exp(t\sqrt[4]{g}\not\nabla\not\nabla\frac{1}{\sqrt[4]{g}})|y >\right]\Bigg|_{x=y} \quad , \tag{80}$$

we can express the local Lorentz anomaly and the Weyl anomaly as

$$\begin{aligned} \ln J_{LL} &= \int d^2x\frac{i}{4}\alpha_{ab}(x)\epsilon_{ab}K(x), \\ \ln J^{2d}_{Dirac} &= \int d^2x\alpha(x)K(x). \end{aligned} \tag{81}$$

This relation is due to the simplicity of 2 dim. It is not valid for higher dimensions. From the above result the Weyl anomaly of two dimensional Dirac field is obtained by replacing $\frac{i}{4}\alpha_{ab}(x)\epsilon_{ab}$ by $\alpha(x)$ in two dimensional local Lorentz anomaly (68). So we can obtain the Weyl anomaly of the Dirac fields as

$$\ln J^{2d}_{Dirac} = \int d^2x\ \alpha(x)\frac{1}{4\pi}\frac{1}{6}\sqrt{g}R \quad . \tag{82}$$

Next, we note the relation of the local Lorentz anomaly and general coordinate anomaly pointed out by [25]. When we set $A^{ab}(x)$ and $A_\mu(x)$ as follows:

$$A_{LL}(x) = \alpha_{ab}(x)A^{ab}(x) \quad , \quad A_{cov}(x) = \xi^\mu(x)A_\mu(x) \quad , \tag{83}$$

then the following relation is satisfied:

$$\nabla^\mu[e_\mu{}^a(x)e_\nu{}^b(x)A^{ab}(x)] = -\frac{1}{2}A_\nu(x). \tag{84}$$

This relation can be assured in this case explicitly. It is valid for higher dimensions.

In Sec.5 of ref.[6], the 2 dim gravitational anomaly was calculated in the light-cone gauge for the external gravity. The counter-term is treated and the Weyl anomaly is obtained in a different way from the present one. The present approach does not fix the gauge, which is desirable because the gravity is not quantized.

7 Chiral Anomaly in Flat and Gravitational Theories

7.1 Chiral U(1) Anomaly in 4 Dim QED

We take a simple model of Euclidean QED.

$$
\begin{aligned}
&\mathcal{L} = \bar{\psi} i\gamma^a D_a \psi - m\bar{\psi}\psi - \tfrac{1}{4}F_{ab}F^{ab} \quad , \\
&F_{ab} = \partial_b A_a - \partial_a A_b \quad , \quad D_a = \partial_a + ieA_a \quad , \quad a = 1, \cdots, 4 \quad ,
\end{aligned}
\tag{85}
$$

where ψ and A_a are Dirac fermion and the gauge(photon) field respectively. The gamma matrices γ^a in this subsection are the same as $\overset{\circ}{\gamma}{}^a$ defined in App.A. The chiral U(1) transformation is given by

$$
\psi' = e^{i\alpha(x)\gamma_5}\psi \quad , \quad \bar{\psi}' = \bar{\psi}\ e^{i\alpha(x)\gamma_5} \quad ,
\tag{86}
$$

Then the anomaly measure and its regularization are given as

$$
\begin{aligned}
&\ln\ J^{QED}_{chiral} = \ln\ \frac{\partial(\psi'(x), \bar{\psi}'(x))}{\partial(\psi(y), \bar{\psi}(y))} = \ 2i\ \mathrm{Tr}\{\alpha(x)\gamma_5\delta^4(x-y)\} + O(\alpha^2) \quad , \\
&I_4\delta^4(x-y) = \lim_{t\to+0} G(x,y;t) \quad , \quad G(x,y;t) \equiv < x|e^{-t(-D^2)}|y > \quad .
\end{aligned}
\tag{87}
$$

The ingredients for the anomaly formula (35) are given by

$$
\begin{aligned}
&(\tfrac{\partial}{\partial t} + (-D^2))\ G(x,y;t) = 0 \quad , \\
&-D^2 \equiv -D_aD_a = -\partial_a\partial_a - V \quad , \quad V = N_a\partial_a + M \quad , \\
&N_a = 2ieA_a \quad , \quad M = -e^2A_aA_a + ie\partial_aA_a - \tfrac{ie}{4}[\gamma^a, \gamma^b]F_{ab} \quad .
\end{aligned}
\tag{88}
$$

From the anomaly formula (35), the chiral anomaly is given by

$$
\begin{aligned}
\frac{1}{2}\ \frac{d}{d\alpha(x)}\ \ln\ J^{QED}_{chiral}\Big|_{\alpha=0} &= \ \mathrm{tr}\ i\gamma_5 G_2^{flat}(x,x;t)|_{t^0} = \frac{1}{(4\pi)^2}\mathrm{tr}(i\gamma_5\frac{1}{2}M^2) \\
&= \frac{i}{(4\pi)^2}\frac{e^2}{2}F_{ab}\tilde{F}^{ab} \quad , \quad \tilde{F}^{ab} \equiv \epsilon^{abcd}F_{cd}
\end{aligned}
\tag{89}
$$

7.2 Chiral U(1) Anomaly in 4 Dim Fermion-Gravity System and Relation to 2 Dim Gravitational Anomaly

In this subsection, we consider the 4 dim gravitational chiral $U(1)$ anomaly [26, 27, 28, 29, 30, 31] and its connection to two dimensional gravitational anomaly in the context of our formalism. The Lagrangian (41) is classically invariant under the following gravitational chiral transformation of the Dirac fermion :

$$
\tilde{\psi}(x)' = e^{i\alpha(x)\gamma_5}\tilde{\psi}(x) \quad , \quad \tilde{\bar{\psi}}(x)' = \tilde{\bar{\psi}}(x)e^{i\alpha(x)\gamma_5}.
\tag{90}
$$

The chiral anomaly is computed by the same procedure as in Sec.2 and Sec.5 and is given by

$$\ln J_{chiral} = -2i \lim_{t\to +0} \mathrm{Tr}[\alpha(x)\gamma_5 G(x,y;t)], \tag{91}$$

where $G(x,y;t)$ is given by (46). We do not explain the detailed calculation because it is the same procedure as in the previous sections. We obtain the chiral anomaly as

$$\begin{aligned}\ln J_{chiral} &= -\frac{1}{(4\pi)^2}\frac{i}{12}\int d^4x\alpha(x)\epsilon^{\lambda\rho\sigma\tau}(\partial_\lambda\partial_\nu f_{\mu a}\delta_{\rho a} + \partial_\lambda\partial_\nu f_{\rho a}\delta_{\mu a}) \\ &\quad \times[\partial_\sigma\partial_\nu f_{\mu b}\delta_{\tau b} + \partial_\sigma\partial_\nu f_{\tau b}\delta_{\mu b} - (\mu \leftrightarrow \nu)] + O(f^2) \\ &= -\frac{1}{(4\pi)^2}\frac{i}{24}\int d^4x\alpha(x)\sqrt{g}e^{\lambda\rho\sigma\tau}R_{\mu\nu\lambda\rho}R^{\mu\nu}{}_{\sigma\tau},\end{aligned} \tag{92}$$

where $e^{\lambda\rho\sigma\tau} = g^{-1/2}\epsilon^{\lambda\rho\sigma\tau}$, and $\epsilon^{\lambda\rho\sigma\tau}$ is the antisymmetric unit tensor in 4 dim.

We investigate the relation between 2 dim gravitational anomaly and 4 dim chiral anomaly suggested in (73)[5, 6]. In order to reduce 4 dim quantities to 2 dim ones, we divide 4 dim expression to $2+2$ dim expression. We can express 4 dim gamma matrices by 2 dim ones:

$$\begin{aligned}\overset{\circ}{\gamma}_{(4)}{}^1 &= \overset{\circ}{\gamma}_{(2)}{}^1 \otimes \sigma^3 \ , \ \overset{\circ}{\gamma}_{(4)}{}^2 = \overset{\circ}{\gamma}_{(2)}{}^2 \otimes \sigma^3 \ , \ \overset{\circ}{\gamma}_{(4)}{}^3 = 1\otimes\sigma^1 \ , \ \overset{\circ}{\gamma}_{(4)}{}^4 = 1\otimes\sigma^2 \ , \\ \overset{\circ}{\gamma}_{(4)5} &= -i\overset{\circ}{\gamma}_{(2)}{}^1\overset{\circ}{\gamma}_{(2)}{}^2 \otimes\sigma^3 = \overset{\circ}{\gamma}_{(2)5}\otimes\sigma^3,\end{aligned} \tag{93}$$

where $Q_{(4)}$ is 4 dim quantity and $Q_{(2)}$ is 2 dim one and σ^i are the Pauli matrices. We consider the special background gravitational field as in the Kaluza-Klein type dimensional reduction. Let us take the case that the spin connection $\omega_\mu{}^{ab}$ is given by the following equation:

$$\omega_{(4)}{}_\mu{}^{ab}(x) = \begin{cases} \omega_{(2)}{}_\mu{}^{ab}(\hat{x}) & \text{if } \mu = 1,2 \text{ and } a,b = 1,2, \\ -i\xi_\mu(\hat{x}) & \text{if } \mu = 1,2 \text{ and } a = 3,\ b = 4, \\ i\xi_\mu(\hat{x}) & \text{if } \mu = 1,2 \text{ and } a = 4,\ b = 3, \\ 0 & \text{otherwise,} \end{cases} \tag{94}$$

$$(x) = (x^1,\cdots,x^4) \quad , \quad (\hat{x}) = (x^1,x^2) \quad . \tag{95}$$

Using the relations

$$\begin{aligned}\overset{\circ}{\sigma}_{(4)}{}^{ab} &= \overset{\circ}{\sigma}_{(2)}{}^{ab}\otimes 1 \quad \text{if } a,b = 1,2, \qquad \overset{\circ}{\sigma}_{(4)}{}^{34} = \frac{1}{2}1\otimes i\sigma^3, \\ \overset{\circ}{\sigma}_{(4)}{}^{43} &= -\frac{1}{2}1\otimes i\sigma^3, \qquad \text{etc.,}\end{aligned} \tag{96}$$

we obtain

$$\begin{aligned}\omega_{(4)}{}_\mu &= \omega_{(2)}{}_\mu{}^{bc}(\overset{\circ}{\sigma}_{(2)}{}^{bc}\otimes 1) + \xi_\mu(1\otimes\sigma^3) \quad \text{for } \mu = 1,2 \quad , \\ \omega_{(4)3} &= \omega_{(4)4} = 0 \quad .\end{aligned} \tag{97}$$

It is sufficient that we consider the correspondence of the operators $\vec{D}_x$ in two theories. So we expand $\vec{D}_{(4)x} = (\sqrt[4]{g}\not\nabla\not\nabla\frac{1}{\sqrt[4]{g}})_{(4)}$ to $2+2$ dimensional expression under Eq.(94) as

$$(\not\nabla\not\nabla)_{(4)} = \left(\sum_{\mu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\mu}\nabla_{\mu(2)}\right)\left(\sum_{\nu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\nu}\nabla_{\nu(2)}\right)\otimes 1 + 1\otimes\left(\sum_{\mu=3}^{4}\partial_{\mu}{}^{2}\right)$$
$$+\left(\sum_{\mu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\mu}\nabla_{\mu(2)}\right)\left(\sum_{\nu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\nu}\xi_{\nu}\right)\otimes\sigma^{3} + \left(\sum_{\mu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\mu}\xi_{\mu}\right)\left(\sum_{\nu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\nu}\nabla_{\nu(2)}\right)\otimes\sigma^{3}$$
$$+\left(\sum_{\mu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\mu}\xi_{\mu}\right)\left(\sum_{\nu=1}^{2} \overset{\circ}{\gamma}_{(2)}{}^{\nu}\xi_{\nu}\right)\otimes 1, \quad (98)$$

For simplicity, we make a calculation by the symbolical expression $G(x,y;t) \equiv \; < x|e^{-t\vec{D}}|y >$ of the solution in the heat equation. If we define

$$A_{chi}(x,y;t) \equiv \mathrm{tr}[\overset{\circ}{\gamma}_{(4)5}G(x,y;t)], \quad (99)$$

then the chiral anomaly is expressed as

$$\ln J_{chiral} = \lim_{t\to+0}\int d^4x\{-2i\alpha(x)A_{chi}(x,x;t)\}. \quad (100)$$

Thus,from (98), we can express $A_{chi}(x,y;t)$ as

$$A_{chi}(x,y;t) \;=\; \mathrm{tr}[(\overset{\circ}{\gamma}_{(2)5}\otimes\sigma^3) < x|\exp\{\;t\;\sqrt[4]{g}[1\otimes\sum_{\mu=3}^{4}\partial_{\mu}{}^{2}$$
$$+(\not\nabla_{(2)}\not\nabla_{(2)}+\not\xi\not\xi)\otimes 1 + (\not\nabla_{(2)}\not\xi+\not\xi\not\nabla_{(2)})\otimes\sigma^3]\frac{1}{\sqrt[4]{g}}\}|y>]. \quad (101)$$

where $\not\nabla_{(2)} = \overset{\circ}{\gamma}_{(2)}{}^{\mu}\nabla_{(2)\mu}, \not\xi = \overset{\circ}{\gamma}_{(2)}{}^{\mu}\xi_{\mu}$. Now we focus on the first-order in ξ^{μ} .

$$A_{chi}(x,y;t)|_{(\xi^{\mu})^1} = t\times\mathrm{tr}_{(2)}[(\overset{\circ}{\gamma}_{(2)5} < x|\sqrt[4]{g}(\not\nabla_{(2)}\not\xi+\not\xi\not\nabla_{(2)})\frac{1}{\sqrt[4]{g}}$$
$$\times\exp[t\sqrt[4]{g}\not\nabla_{(2)}\not\nabla_{(2)}\frac{1}{\sqrt[4]{g}}]|y>]\otimes\mathrm{tr}_{(2)} < x|\exp[t\sum_{\mu=3}^{4}\partial_{\mu}{}^{2}]|y> \quad . \quad (102)$$

Using the relation

$$\mathrm{Tr}_{(2)} < x|\exp[t\sum_{\mu=3}^{4}\partial_{\mu}{}^{2}]|y> = \mathrm{tr}(\frac{1}{4\pi t}I_2) = \frac{1}{2\pi t} \quad , \quad (103)$$

and the expression for $A_{cov}(x)$, (70), the equation (102) says

$$\lim_{t\to+0}A_{chi}(x,x;t)|_{(\xi^{\mu})^1} = \frac{1}{2\pi}\times 2A_{cov}(x), \quad (104)$$

for the background gravitational field (94).

Using this relation, we can calculate gravitational anomaly $A_{cov}(x)$ from the chiral anomaly $A_{chi}(x,x;t)$. For (94), the curvature tensor becomes

$$R_{(4)\mu\nu}{}^{ab} = \begin{cases} R_{(2)\mu\nu}{}^{ab} & \text{if } \mu,\nu = 1,2 \text{ and } a,b = 1,2, \\ (-i\nabla_\mu\xi_\nu + i\nabla_\nu\xi_\mu) & \text{if } \mu,\nu = 1,2 \text{ and } a = 3,\ b = 4, \\ (i\nabla_\mu\xi_\nu - i\nabla_\nu\xi_\mu) & \text{if } \mu,\nu = 1,2 \text{ and } a = 4,\ b = 3, \\ 0 & \text{otherwise} \end{cases} \tag{105}$$

If we substitute the above expression into $A_{chi}(x,x;t)$-part of (92), we obtain

$$\begin{aligned} A_{cov}(x) &= \pi \lim_{t\to+0} A_{chi}(x,x;t)|_{(\xi^\mu)^1} \\ &= \frac{1}{4\pi}\frac{1}{48}\epsilon^{ab34} R_{(2)}{}^{\mu\nu}{}_{ab}(-i\nabla_\mu\xi_\nu + i\nabla_\nu\xi_\mu) \times 2 \\ &= -\frac{1}{4\pi}\frac{i}{12}\nabla_\mu\xi_\nu R_{(2)}{}^{\mu\nu}{}_{ab}\epsilon^{ab} \\ &= -\frac{1}{4\pi}\frac{i}{12}\sqrt{g}e^{\nu\mu}\xi_\nu\nabla_\mu R + [\text{tot. div}], \end{aligned} \tag{106}$$

where the [tot. div] is the total divergence term. This term does not contribute the gravitational anomaly, so it turns out to give the same result as in the Sect. 6.2.

Generally, we can extend above discussion to the higher dimension. The following relations are satisfied in the solutions, $G_l^{(n)}(x,y;t)$, of the heat equation:

$$\frac{1}{4\pi t}G_l^{(4k)}(x,y;t)|_{t^m} = G_l^{(4k-2)}(x,y;t)|_{t^{m+1}}, \tag{107}$$

whence $4k-2$ dimensional gravitational anomaly is related to $4k$ dimensional chiral anomaly. We can connect two anomalies by the similar discussion in this section.

As demonstrated in Sect.6 and 7, the presented anomaly formulae provide a powerful tool to calculate all kinds of anomalies concretely. This is complementary to ref.[6] where the chiral anomaly is derived from the result of the differential geometry and the gravitational anomaly is indirectly derived using the relation between the two anomalies.

8 Discussions

We have presented a systematic approach to general anomaly calculation. It is based on the propagator theory. Fujikawa's general standpoint about anomaly is taken. In the evaluation of the Jacobian due to the change of the path-integral measure, we take the heat-kernel regularization. The 2 dim and 4 dim general anomaly formulae are obtained. The known various anomalies and some relations between them are explicitly derived from the formulae. We have also presented the graphical technique

for the representation of (global SO(n)) invariants and covariants appearing in the weak field expansion calculation. For the flat space(-time) case, the 4 dim anomaly formula reduces to the 'tHooft's 1-loop counter-term formula except total derivative terms.

We make some additional comments and refer to the future directions.

1. The present systematic calculation and the graph technique enable us to do the anomaly calculation using a computer. It opens the possibility to calculate anomalies in other interesting theories such as Weyl anomaly of the gravitational theories, in higher dimensions, coupled with various matter fields.

2. The anomaly calculations so far are mainly concerned with the matter (1-loop) quantum effect. The effect due to gauge or gravity quantum excitation is also important, especially for the Weyl anomaly[32, 33]. When the present approach is combined with the background field method, matter fields and gauge or gravity fields can be equally treated. The generalization to such a direction is interesting.

3. The higher-loop effects are important for the Weyl anomaly, because it is directly related to the renormalization-group β-function. The higher-loop generalization of the present approach is also interesting.

4. Recently (4 dim) super Yang-Mills theory is vigorously investigated in relation to the dual symmetry[34, 35]. In the problem some anomalies, such as Konishi anomaly[36] and the super-Weyl anomaly, play an important role. We hope the present approach helps to clarify some part of this interesting problem.

Renormalization and anomaly are two outstandingly important aspects of quantum field theory. They have been giving us various rich information and helping us to understand the field theory. The direct relation between the 1-loop counter-term formula and the anomaly formula clearly shows the intimate relation between them.

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Appendix A. Notations and Some Useful Formulae

Here we collect the present notation and some useful formulae.

A.1 Basic Notation

$$
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \quad , R^{\lambda}{}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\tau\nu}\Gamma^{\tau}_{\mu\sigma} - \nu \leftrightarrow \sigma \quad ,
$$
$$
R_{\mu\nu} = R^{\lambda}{}_{\mu\nu\lambda} \quad , \quad R = g^{\mu\nu} R_{\mu\nu} \quad , \quad g = +\mathrm{det} g_{\mu\nu} \quad . \tag{108}
$$

The covariant derivative for a vector field A_μ:

$$
\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^{\lambda}{}_{\mu\nu} A_\lambda \quad . \tag{109}
$$

The covariant derivative for general tensor field $T_\Sigma = T^{\sigma_1 \cdots \sigma_r}_{\tau_1 \cdots \tau_s}$:

$$
\nabla_\mu T_\Sigma = \partial_\mu T_\Sigma + \Gamma^{\lambda}{}_{\mu\nu} [T_\Sigma]^{\nu}{}_{\lambda}, \tag{110}
$$

where

$$
[T^{\sigma_1 \cdots \sigma_r}_{\tau_1 \cdots \tau_s}]^{\nu}{}_{\lambda} = \sum_{p=1}^{r} \delta^{\sigma_p}{}_{\lambda} T^{\sigma_1 \cdots \sigma_{p-1} \nu \sigma_{p+1} \cdots \sigma_r}_{\tau_1 \cdots \tau_s} - \sum_{q=1}^{s} \delta^{\nu}{}_{\tau_q} T^{\sigma_1 \cdots \sigma_r}_{\tau_1 \cdots \tau_{q-1} \lambda \tau_{q+1} \cdots \tau_s}. \tag{111}
$$

A.2 Weak-Field Expansion

$$
\begin{aligned}
g_{\mu\nu} &= \delta_{\mu\nu} + h_{\mu\nu} \quad , \\
R_{\mu\nu\lambda\sigma} &= \frac{1}{2}(\partial_\lambda \partial_\nu h_{\mu\sigma} - \partial_\lambda \partial_\mu h_{\nu\sigma} - \lambda \leftrightarrow \sigma) + O(h^2) \quad , \\
R_{\mu\nu} &= \frac{1}{2}(\partial_\mu \partial_\nu h - \partial_\mu \partial_\alpha h_{\alpha\nu} - \partial_\nu \partial_\alpha h_{\alpha\mu} + \partial^2 h_{\mu\nu}) + O(h^2) \quad , \\
R &= \partial^2 h - \partial_\alpha \partial_\beta h_{\alpha\beta} + O(h^2) \quad , \\
\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) + O(h^2) \quad , \\
\sqrt{g} &= 1 + \frac{1}{2} h_{\mu\mu} + O(h^2) \quad .
\end{aligned} \tag{112}
$$

A.3 Integral Formula

$$
\begin{aligned}
\int d^n w\ e^{-Aw^2} &= (\frac{\pi}{A})^{\frac{n}{2}} \quad , \\
\int d^n w\ w^\mu w^\nu e^{-Aw^2} &= \frac{\delta^{\mu\nu}}{2}(\frac{\pi}{A})^{\frac{n}{2}}\frac{1}{A} \quad , \\
\int d^n w\ w^\mu w^\nu w^\lambda w^\sigma e^{-Aw^2} &= \frac{[\mu\nu\lambda\sigma]}{4}(\frac{\pi}{A})^{\frac{n}{2}}\frac{1}{A^2} \quad , \\
\int d^n w\ w^{\mu_1}\cdots w^{\mu_6} e^{-Aw^2} &= \frac{[\mu_1\cdots\mu_6]}{8}(\frac{\pi}{A})^{\frac{n}{2}}\frac{1}{A^3} \quad , \\
\int d^n w\ w^{\mu_1}\cdots w^{\mu_{2s}} e^{-Aw^2} &= \frac{[\mu_1\cdots\mu_{2s}]}{2^s}(\frac{\pi}{A})^{\frac{n}{2}}\frac{1}{A^s} \quad , \qquad (113)
\end{aligned}
$$

where

$$
\begin{aligned}
[\mu\nu\lambda\sigma] &\equiv \delta^{\mu\nu}\delta^{\lambda\sigma} + \delta^{\mu\sigma}\delta^{\nu\lambda} + \delta^{\mu\lambda}\delta^{\nu\sigma} \quad , \\
[\mu_1\cdots\mu_6] &\equiv \delta^{\mu_1\mu_2}[\mu_3\cdots\mu_6] + \delta^{\mu_1\mu_3}[\cdots] + \cdots + \delta^{\mu_1\mu_6}[\cdots] \quad , \\
[\mu_1\cdots\mu_{2s}] &\equiv \delta^{\mu_1\mu_2}[\mu_3\cdots\mu_{2s}] + \delta^{\mu_1\mu_3}[\cdots] + \cdots + \delta^{\mu_1\mu_{2s}}[\cdots] \quad . \qquad (114)
\end{aligned}
$$

$$
\begin{aligned}
G_0(x;t) &= \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}}\ I_N \quad , \\
\int d^n x\ x^{\mu_1}\cdots x^{\mu_{2s}} G_0(x;t) &= [\mu_1\cdots\mu_{2s}](2t)^s\ I_N \quad , \\
\int_0^1 dr\ r^p(1-r)^q &= \frac{p!\ q!}{(p+q+1)!} \quad . \qquad (115)
\end{aligned}
$$

A.4 Spinor and Vielbein

$$
\begin{aligned}
&\{\overset{\circ}{\gamma}{}^a, \overset{\circ}{\gamma}{}^b\} = 2\delta^{ab} \quad ,(\overset{\circ}{\gamma}{}^a)^\dagger = \overset{\circ}{\gamma}{}^a \quad ,\overset{\circ}{\sigma}{}^{ab} = \frac{1}{4}[\overset{\circ}{\gamma}{}^a, \overset{\circ}{\gamma}{}^b] = -\overset{\circ}{\sigma}{}^{ba} \quad , \\
&\gamma^\mu = e^\mu{}_a \overset{\circ}{\gamma}{}^a \quad ,\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \\
&\bar\psi = \psi^\dagger\gamma^4 \quad . \qquad (116)
\end{aligned}
$$

$$
\omega_\mu{}^a{}_b = (\Gamma^\lambda{}_{\mu\nu} e_\lambda{}^a - \partial_\mu e_\nu{}^a) e^\nu{}_b \quad ,\omega_\mu = \frac{1}{2}\overset{\circ}{\sigma}{}^{ab}\omega_{\mu ab} \quad . \qquad (117)
$$

$$
\begin{aligned}
&\nabla_\mu\psi = (\partial_\mu + \omega_\mu)\psi \quad ,\bar\psi\overleftarrow{\nabla}_\mu = \bar\psi(\overleftarrow{\partial}_\mu - \omega_\mu), \\
&\nabla\!\!\!/ = \gamma^\mu\nabla_\mu \quad ,\bar\psi\overleftrightarrow{\nabla\!\!\!/}\psi = \bar\psi(\gamma^\mu\nabla_\mu\psi) - (\bar\psi\gamma^\mu\overleftarrow{\nabla}_\mu)\psi \quad . \qquad (118)
\end{aligned}
$$

In 4 dim

$$
\begin{aligned}
&\overset{\circ}{\gamma}{}_5 = -\overset{\circ}{\gamma}{}^1\overset{\circ}{\gamma}{}^2\overset{\circ}{\gamma}{}^3\overset{\circ}{\gamma}{}^4 \quad ,(\overset{\circ}{\gamma}{}_5)^2 = 1, \\
&e^{\lambda\rho\sigma\tau} = g^{-1/2}\epsilon^{\lambda\rho\sigma\tau} \quad ,\epsilon^{1234} = +1 \quad . \qquad (119)
\end{aligned}
$$

where $\epsilon^{\lambda\rho\sigma\tau}$ is the antisymmetric unit tensor in four dimension.

In 2 dim

$$
\begin{aligned}
&\overset{\circ}{\gamma}_5 = -i\overset{\circ}{\gamma}{}^1\overset{\circ}{\gamma}{}^2 \quad ,(\overset{\circ}{\gamma}_5)^2 = 1 \quad , \\
&\overset{\circ}{\sigma}{}^{ab} = \frac{i}{2}\overset{\circ}{\gamma}_5\epsilon^{ab} \quad ,e^{\lambda\rho} = g^{-1/2}\epsilon^{\lambda\rho} \quad ,\epsilon^{12} = +1 \quad .
\end{aligned}
\tag{120}
$$

Appendix B. Graphical Representation of $\partial_\mu\partial_\nu h_{\alpha\beta}$ and $(\partial\partial h)^2$-Invariants

In the weak field expansion of n-dim Euclidean gravity : $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, |h| \ll 1$, we must generally treat global SO(n)-tensors which are composed of $h_{\mu\nu}$, $\partial_\alpha h_{\mu\nu}$, $\partial_\alpha\partial_\beta h_{\mu\nu}, \cdots$. Let us focus here on those tensors which are composed only of $\partial_\alpha\partial_\beta h_{\mu\nu}$. We introduce its graphical representation as shown in Fig.2 in subsect.4.4 of the text. Contraction of suffixes is graphically represented by gluing the dotted lines with the same suffix. For example

Fig.6

Independent $\partial\partial h$-scalars (Dimension $(Mass)^2$) are the following two graphs.

Fig.7

In the ordinary literal expression, P=$\partial^2 h$, Q=$\partial_\mu\partial_\nu h_{\mu\nu}$. Similarly we can list up all independent $(\partial\partial h)^2$-scalars (Dimension $(Mass)^4$) as follows. They are grouped by the number of the suffix-loop.

loop no =1

Fig.8

loop no =2

Fig.9

loop no =3

Fig.10

loop no =4

Fig.11

We have totally 13 invariants (10 connected, 3 disconnected). Their literal expressions are listed in Table 2 of the text(subsect.4.4).

As some simple applications, we give the weak-gravity expansion of Riemann

tensors in the graphical way as follows.

(121)

The products of Riemann tensors are similarly expressed, and their $O((\partial\partial h)^2)$-parts are the linear combinations of the above 13 invariants. There are 4 independent invariants of dimension $(Mass)^4$ including total derivatives: $\nabla^2 R$, R^2 , $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\lambda}R^{\mu\nu\lambda\sigma}$. The 13 coefficients for each invariants are listed in Table 4.

Graph	$\nabla^2 R$	R^2	$R_{\mu\nu}R^{\mu\nu}$	$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$
$A1$	1	0	0	-2
$A2$	2	0	$\frac{1}{2}$	0
$A3$	0	0	$\frac{1}{2}$	0
$B1$	-2	0	-1	0
$B2$	2	0	-1	0
$B3$	$-\frac{3}{2}$	0	0	1
$B4$	0	0	0	1
Q^2	0	1	0	0
$C1$	$\frac{1}{2}$	0	$\frac{1}{4}$	0
$C2$	-1	0	$\frac{1}{4}$	0
$C3$	-1	0	$\frac{1}{2}$	0
PQ	0	-2	0	0
P^2	0	1	0	0

Table 4 Weak-Expansion of Invariants with (Mass)^4-Dim. : $(\partial\partial h)^2$-Part

We use the above result in the Weyl anomaly calculation of Sec.4.4 and Sec.5. The results of this appendix B are valid to general dimension of space(-time).

Appendix C. Graphical Representation of Anomaly Formula

In the anomaly formula in 4 dim (Sect.4.3), we must deal with invariants made of two tensors out of $(\partial\partial W, \partial N, M)$. We introduce a useful graphical representation for them. First we define it for the basic elements: $\partial_\alpha\partial_\beta W_{\mu\nu}$ and $\partial_\alpha N_\mu$ as shown in Fig.1 of the text (Sect 4.3).

We notice the symmetry of $\partial_\alpha\partial_\beta W_{\mu\nu}$, with respect to the exchange of suffixes, is the same as that of $\partial_\alpha\partial_\beta h_{\mu\nu}$. Therefore the first group of invariants $(\partial\partial W)^2$ are obtained just by the substitution of a single solid line by a double solid line in the 13 invariants in App.B.

i) $(\partial\partial W)^2$

$$
\begin{array}{lllll}
\bar{A}1, & \bar{A}2, & \bar{A}3, & & \\
\bar{B}1, & \bar{B}2, & \bar{B}3, & \bar{B}4, & \bar{Q}^2, \\
\bar{C}1, & \bar{C}2, & \bar{C}3, & \bar{P}\bar{Q}, & \\
\bar{P}^2 & & & &
\end{array}
$$

Other groups of independent invariants are as follows.

ii) $\partial\partial W \times \partial N$

Fig.12

iii) $\partial N \times \partial N$

Fig.13

iv) $M \times (\partial\partial W, \partial N, M)$

Fig.14

Totally there are 16 connected ones and 10 disconnected ones. We use,in the text, the above notation for 26 independent invariants. The ordinary (literal) expression for each graph is listed in Table 1 in Sect.4.3.

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Figure Captions

- Fig.1 Graphical representation of $\partial_\alpha\partial_\beta W_{\mu\nu}$ and $\partial_\alpha N_\mu$.
- Fig.2 Graphical representation of $\partial_\mu\partial_\nu h_{\alpha\beta}$.
- Fig.3 Graphical representation of $\partial_\mu\partial_\nu {f_\alpha}^a$ ((a))and $\delta_{\mu a}$ ((b)).
- Fig.4 (a) Graphical relation between $\partial_\lambda\partial_\sigma h_{\mu\nu}$ (Fig.2) and $\partial_\lambda\partial_\sigma {f_\mu}^a$ (Fig.3); (b) Graphical definition of 'anti-symmetric metric'.
- Fig.5 Graphical representation of $\partial_\lambda\partial_\sigma W_{\mu\nu},\ \partial_\sigma N_\lambda$ and M for the fermion-gravity system.
- Fig.6 Graphical representation of $\partial_\alpha\partial_\beta h_{\alpha\nu}$.
- Fig.7 Graphical representation of $\partial^2 h$ (P) and $\partial_\mu\partial_\nu h_{\mu\nu}$ (Q).
- Fig.8 Graphical representation of $(\partial\partial h)^2$-scalars. The number of suffix-loops is 1.
- Fig.9 Graphical representation of $(\partial\partial h)^2$-scalars. The number of suffix-loops is 2.
- Fig.10 Graphical representation of $(\partial\partial h)^2$-scalars. The number of suffix-loops is 3.
- Fig.11 Graphical representation of $(\partial\partial h)^2$-scalars. The number of suffix-loops is 4.
- Fig.12 Graphical representation of the anomaly formula : $\partial\partial W\times\partial N$.
- Fig.13 Graphical representation of the anomaly formula : $\partial N\times\partial N$.
- Fig.14 Graphical representation of the anomaly formula : $M\times(\partial\partial W,\ \partial N,\ M)$.